Breaking of a Riemann wave in dispersive hydrodynamics

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A general method is developed for analytically solving Whitham’s modulation equations, which describe the structure of a dissipationless shock wave after an arbitrary monotonic profile breaks in a Korteweg–de Vries hydrodynamics.

1. A simple Riemann wave is described by the equation \( \partial_t r + V(r) \partial_x r = 0 \), which has the solution
\[
x - V(r) t = W(r),
\]
where the function \( W(r) \) is the inverse of the initial profile \( r = r_0(x) \). After a simple wave breaks in a Korteweg–de Vries hydrodynamics, the solution is described by the three functions \( r_1, r_2, r_3 \) (Fig. 1), which satisfy Whitham’s modulation equations
\[
\partial_t r_i + V_i(r) \partial_x r_i = 0.
\]
The characteristic velocities \( V_i(r) \) are given by explicit expressions, and \( r = (r_1, r_2, r_3) \) (Refs. 1–3).

A vector generalization of Riemann wave (1) for system (2) is
\[
x - V_i(r) t = W_i(r),
\]
but the function \( W_i(r) \) is not arbitrary. It must satisfy compatibility conditions found by substituting (3) into (2):
\[
\frac{\partial_i W_j}{W_i - W_j} = \frac{\partial_i V_j}{V_i - V_j}, \quad i \neq j, \quad \partial_i \equiv \partial_{r_i}
\]
[the construction in (3), (4) is known as the generalized hodograph method\(^{3,4}\)]. We would also obtain Eqs. (4) if we analyzed the system of equations
\[
\partial_t r_i + W_i(r) \partial_x r_i = 0,
\]
which specifies the flux \( r_i(\tau) \) which commutes with the solution of (2), i.e., the flux \( r_i(t) \). In other words, the general solution of compatibility equations (4) in \( r \) space describes all the \( r \) fluxes which commute with \( r(t) \) from (2) (Ref. 3).
2. As a result of Eqs. (2), the number of waves is conserved:

\[ \partial_t k + \partial_x (kU) = 0, \] (6)

where \( k \) is the wave number, and \( U \) is the phase velocity. In the Korteweg–de Vries case we would have

\[ U = \frac{1}{3} \sum_{j=1}^{3} r_j, \quad \lambda = \frac{2\pi}{k} = 6^{1/2} \int_{r_1}^{r_2} d\mu \left[ \prod_{j=1}^{3} (\mu - r_j) \right]^{-1/2} = \frac{6^{1/2} K(m)}{(r_3 - r_1)^{1/2}}. \] (7)

Here \( K(m) \) is the complete elliptic integral of the first kind, and \( m = (r_2 - r_1)/(r_3 - r_1) \).\n
Going over to the Riemann variables \( r_j \) in (6), we obtain a “potential” representation for the characteristic velocities:

\[ V_i(r) = U + k \partial_i U / \partial_x k = U - \lambda \partial_i U / \partial_x \lambda. \] (8)

Let us consider the following equation, which commutes with (6):

\[ \partial_t k + \partial_x (kf) = 0, \] (9)

where \( f \) is a generalized phase velocity [certain equations of the type in (9) have the natural meaning of conservation laws for the number of waves for higher-order Korteweg–de Vries situations]. By analogy with (8) we find the following expression for \( W_i \) in terms of the (scalar) function \( f \):

\[ W_i = f - \lambda \partial_i f / \partial_x \lambda = f + (V_i - U) \partial_i f / \partial_x U. \] (10)

Substituting (8) and (10) into Eqs. (4), we find a scalar formulation of the compatibility equations:

\[ \frac{\partial_i^2 f}{\partial_i f - \partial_j f} \frac{\partial_k^2 \lambda}{\partial_i \lambda - \partial_j \lambda}, \quad i \neq j. \] (11)
Using (7), we finally find
\[
\frac{\partial_{ij}^2 f}{\partial_i f - \partial_j f} = \frac{1}{2(r_i - r_j)}
\]
or
\[
E_{ij} f = 0, \quad \text{where} \quad E_{ij} = \partial_{ij}^2 - (\partial_i - \partial_j)/2(r_i - r_j).
\] (12)

Equations (12) were derived by a different method in Ref. 5. Each of these equations, for a given pair, \(i, j\), is an Euler–Poisson equation in \(r_i, r_j\) at a fixed \(r_k = r_k0\) \((k \neq i, j)\). Of importance to the discussion below are the homogeneous solutions of (12) of the form \(f = r^q \Phi (-q, 1/2; 1/2 - q; r_j/r_j)\), where \(q\) is an arbitrary number (not necessarily an integer), and \(\Phi(a, b; c; z)\) is the solution of the corresponding hypergeometric equation. 6

3. Let us examine the problem of the breaking of simple wave (1) with a monotonic initial profile in a dispersive Korteweg–de Vries hydrodynamics. We assume that the breaking begins at \(t = 0\) at the point \(x = 0, r = 0\) (Fig. 1), where
\[
r(x, 0) = \begin{cases} 
  r_0^+(x) < 0 & \text{for } x \geq 0, \\
  r_0^-(x) > 0 & \text{for } x < 0;
\end{cases} \quad W(r) = \begin{cases} 
  W_+(r) & \text{for } r \leq 0, \\
  W_-(r) & \text{for } r > 0.
\end{cases} \quad (13)
\]
The solution which we need describes the evolution of the dissipationless shock wave, which lies between the boundaries \(x = x^- (t)\) (the trailing edge) and \(x = x^+ (t)\) (the leading edge). This solution satisfies certain conditions on the curves of \(x^\pm (t)\). These conditions are that the “external” solution in (1), \(r(x,t)\), join with the solution \((r_1, r_2, r_3)\) of the “internal” modulation equations, (2) (Ref. 2):
\[
r_3(x^-, t) = r_-(x^-, t) \quad \text{for } r_2 = r_1; \quad r_1(x^+, t) = r_+(x^+, t) \quad \text{for } r_2 = r_3. \quad (14)
\]

Interestingly, when we go over to \(r\) space, conditions (14) take the simple form
\[
W_1 = W_+(r_1) \quad \text{for } r_2 = r_3; \quad W_3 = W_-(r_3) \quad \text{for } r_2 = r_1. \quad (15)
\]

As a result, instead of a problem with conditions at an unknown boundary, (14), in \(r\) space, linear system (4) satisfies simple linear conditions at given boundaries. 7 Switching to the scalar function \(f\), and using (10) with \(r_2 = 0\), we find the boundary conditions
\[
f = f_-(r_3) = \frac{1}{2} r_3^{-1/2} \int_0^{r_3} x^{-1/2} W_-(x) dx \quad \text{for } r_1 = r_2 = 0, \quad (16)
\]
\[
f = f_+(r_1) = \frac{1}{2} (-r_1)^{-1/2} \int_0^{-r_1} x^{-1/2} W_+(x) dx \quad \text{for } r_3 = r_2 = 0.
\]
The satisfaction of conditions (16) on the \(r_2 = 0\) plane implies the satisfaction of (15) (for the regular solution). Combining Eqs. (4), we easily find the result
\[ \partial_t W_1 = (\partial_2 + \partial_3) W_1 / r_2 = r_2 \] for arbitrary \( r_1 \). In other words, the function \( W_1 \) is constant along the leading edge (and corresponding comments apply to \( W_3 \) and the trailing edge).

For \( f(r_1, r_2, r_3) \) we thus have system of equations (12) in the region \( r_1 \leq r_2 \leq r_3, r_1 < 0, r_3 > 0 \) between the \( r_2 = r_1 \) and \( r_2 = r_3 \) planes (Fig. 2), with boundary conditions (16). Let us construct a solution for this problem.

We first find \( f(r_1, 0, r_3) \) from Goursat problem (12), (16):

\[ E_{31} f = 0, \quad f|_{r_1=0} = f-(r_3), \quad f|_{r_3=0} = f+(r_1). \] (17)

Since problem (17) is linear, it is sufficient to solve it for the case (for example) \( f_+(r_1) = 0 \). From the representation of \( f-(r_3) \) as a Mellin integral,

\[ f-(r_3) = \frac{1}{2\pi i} \int_{c-iz}^{c+iz} r_3^q S(q) dq, \quad S(q) = \int_0^\infty r_3^{-q-1} f-(r_3) dr_3, \]

it is clear that we can restrict the analysis to \( f^{(q)}(r_3) = r_3^q \). A solution of this problem can be written out explicitly:

\[ f^{(q)}(r_1, r_3) = r_3^q \left[ \frac{\Gamma(q+1)}{\Gamma(q+1/2)} \right]^{1/2} \frac{1}{q+1/2} u_4(r_1/r_3), \] (18)

where \( u_4(z) = z^{-1/2} F(q + 1, 1/2; q + 3/2; z^{-1}) \) is the corresponding Kummer solution of the hypergeometric equation with the parameters \( (-q, 1/2; 1/2 - q) \), and \( F(a, b; c; z) \) is the Gauss hypergeometric function.

The solution found, \( f(r_1, 0, r_3) \), can then be thought of as a condition (of the Goursat type) for corresponding boundary-value problems in the \( r_1 = \text{const}, r_2 > 0 \)
and \( r_3 = \text{const}, \ r_2 > 0 \) planes (Fig. 3). In planes with \( r_1 = \text{const} \), for example, we have the following problem \( (r_1 \text{ appears as a parameter; } r_2 > 0) \): We have \( E_{3} f^{*}(r_2, r_3) = 0 \), \( f^{*}(r_1, 0, r_3) = f^{*}(q)(r_1, r_3) \) [see (18)], and \( f^{*}(r_1, r_2, r_3) \) is regular at \( r_2 = r_3 \). By virtue of the linearity of the problem, it is again convenient to write \( f^{*}(q)(r_1, r_3) \) as a Mellin integral (or as a power series in \( r_3 \)). The solution for each term has form like that of (18), with \( u_4(z) \) replaced by \( u_5(z) \), which is regular at the leading edge \( (z = 1) \). A solution is found in the planes \( r_3 = \text{const}, \ r_2 < 0 \) in a corresponding way. The solution is thus constructed on the two sides of the \( r_2 = 0 \) plane, at which the function \( f \) is given. The continuity of the normal derivative \( \partial f / \partial t \) at \( r_2 = 0 \) can be checked directly (cf. Ref. 2). The \( x^\pm (t) \) curves, which bound the region of the dissipationless shock wave, are found through a joint analysis of solution (3) and the conditions \( dx^\pm / dt = V^\pm \) at the boundaries \( (V^\pm \) are multiple characteristic velocities).²

4. Examples. It is clear from the discussion above that, without any loss of generality, it is sufficient to analyze the breaking problem with initial data (13), where

\[
\begin{align*}
    r_0^-(x) &= (-x)^{1/q_-}, \quad r_0^+(x) = -x^{1/q_+}; \quad W_-(r) = -r^{q_-}, \quad W_+(r) = (-r)^{q_+}, \quad q_\pm > 1.
\end{align*}
\]

With \( q_+ = q_- = q \), the solutions \( r_i(x, t) \) which we need are self-similar:\(2,3 \) \( r_i = t^\gamma \int_i(x/t^{\gamma+1}), \) where \( \gamma = 1/(q - 1) \).

a) A quasisimple wave \( r = (0, r_2, r_3) \). We ultimately find a family of solutions with \( r_1 = 0 \), which were discussed in Ref. 7. A wave of this sort is described by the equation \( E_{3} f^{*}(r_2, r_3) = 0 \) with given \( f(0, r_3) = -r^{q}/(2q + 1) \), corresponding to a breaking of a monotonic profile \( R_0^-(x) = (-x)^{1/2}, \ R_0^+(x) = 0. \) The solution which we need, and which is regular on the bisector \( r_2 = r_3 \), is \( f(r) = -r^{q/2} \Gamma(1 + q)u_5(r_2/r_3)/2\Gamma(q + 3/2) \). For integer values \( q = M \), the hypergeometric series is truncated, and the solution takes the symmetric form

\[
f(r) = P_M(r_2, r_3) = -\frac{2^M M!}{(2M - 1)!!(2M + 1)} \sum_{k_2 + k_3 = M} \frac{(1/2)k_2 (1/2)k_3 r_2^{k_2} r_3^{k_3}}{k_2! k_3!},
\]

\( (a)_n = \Gamma(a + n)/\Gamma(a) \).
b) Breaking of an antisymmetric profile \( r = (r_1, r_2, r_3) \). We assume \( q = M \) (an integer). Then for odd values of \( M \) we have

\[
f(r) = P_M(r) = -\frac{2^M M!}{(2M - 1)!!(2M + 1)} \sum_{k_1 + k_2 + k_3 = M} \frac{(\frac{1}{2})_{k_1} (\frac{1}{2})_{k_2} (\frac{1}{2})_{k_3}}{k_1!k_2!k_3!} r_1^{k_1} r_2^{k_2} r_3^{k_3},
\]

and for even values of \( M \) we have

\[
f(r) = \begin{cases} 
- P_M(r) + I(-r_1, -r_2, r_3) & \text{for } r_2 < 0 \\
 P_M(r) - I(r_3, r_2, -r_1) & \text{for } r_2 > 0.
\end{cases}
\]

Although an explicit integral representation of \( I(r) \) is known, it is lengthy, and we will not reproduce it here. For even integer values of \( q \), the solution is thus not a polynomial solution. As we mentioned earlier, the derivative \( \partial_f(r_2 = 0) \) is continuous. Nevertheless, the \( r_2 = 0 \) plane is obviously singular. This result is not surprising: An initial profile with even \( q \) is not an analytic function, and it cannot be found as a result of the evolution of a smooth profile of a Riemann wave. Only a profile with odd values of \( q \), for which solution (20) is of a polynomial nature and has no singularities, satisfies the evolution properties.

5V. R. Kudashev and S. E. Sharapov, Preprint IAÉ-5221/6, I. V. Kurchatov Institute of Atomic Energy, Moscow, 1990.

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