Dispersive shock waves in photorefractive media

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References

Dispersive Shock Waves (DSWs) (a.k.a. undular bores or collisionless shocks)

**DSW:** the dispersive wave-train transition between two different basic states (dispersive resolution of a large-scale nonlinear wave-breaking).

**Internal structure:** modulated nonlinear periodic wave transforming into a solitary wave (bright or dark) at one edge and into a vanishing amplitude wave-packet at the opposite edge.

Important: the speeds of the soliton and harmonic edges $s_{1,2}$ are different, $s_1 \neq s_2$, and, as a result, the classical shock jump conditions for the depth and horisontal velocity are not applicable.

Below we consider only purely conservative DSWs (no dissipation)!
DSWs in a photorefractive crystal: an experiment

Transformation of input profile of light intensity into its output profile as a function of crystal’s length $z$
The light beam propagation is described by the nonlinear wave equation

\[ i \frac{\partial \psi}{\partial z} + \frac{1}{2k_0} \Delta_\perp \psi + \frac{k_0}{n_0} \delta n (|\psi|^2) \psi = 0, \tag{1} \]

Here \( \psi \) is the complex amplitude of the electromagnetic wave with the wave number \( k_0 = 2\pi n_0 / \lambda \); \( z \) is the coordinate along the beam, \( x, y \) are transverse coordinates, \( r = (x, y) \), \( \Delta_\perp = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \) is the transverse Laplacian, \( n_0 \) is the linear refractive index.

In a photo-refractive medium with local saturable nonlinearity we have

\[ \delta n = -\frac{1}{2} n_0^3 r_{33} E_p \frac{\rho}{\rho + \rho_d}, \tag{2} \]

where \( E_p \) is the electric field applied to the crystal, \( r_{33} \) electro-optical index, \( \rho = |\psi|^2 \), and \( \rho_d \) is the saturation intensity.
The Nonlinear Schrödinger Equation with local saturable nonlinearity

After introducing dimensionless variables we arrive at the generalised nonlinear Schrödinger (GNLS) equation

\[ i \frac{\partial \psi}{\partial z} + \frac{1}{2} \Delta_\perp \psi - f(|\psi|^2)\psi = 0, \]  

(3)

where for a saturable photorefractive medium we have

\[ f(\rho) = \frac{\rho}{1 + \gamma \rho}. \]  

(4)

In case of a single transverse space coordinate \( x \) we have

\[ i \frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{|\psi|^2}{1 + \gamma |\psi|^2} \psi = 0. \]  

(5)

If the saturation effects are negligibly small, \( \gamma |\psi|^2 \ll 1 \), then Eq. (5) reduces to the standard integrable cubic NLS equation with defocussing

\[ i \frac{\partial \psi}{\partial z} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} - |\psi|^2 \psi = 0. \]  

(6)
“Fluid dynamics” representation

We use the (Madelung) transformation

$$\psi(r, z) = \sqrt{\rho} \exp(i\theta), \quad u = \nabla_{\perp} \theta,$$  \hspace{1cm} (7)

to represent the GNLS equation in the fluid dynamics-like form

$$\rho_{z} + \nabla_{\perp}(\rho u) = 0,$$

$$u_{z} + (u \nabla_{\perp} u) + \nabla_{\perp} f(\rho) - \nabla_{\perp} \left[ \frac{\Delta_{\perp} \rho}{4\rho} - \frac{(\nabla_{\perp} \rho)^{2}}{8\rho^{2}} \right] = 0,$$  \hspace{1cm} (8)

Note: vorticity is ruled out by (7). Also $\rho \geq 0$. 
Two canonical fluid dynamics problems involving formation of DSWs

- Dispersive (NLS) Riemann problem (1D)

- Supersonic (NLS) flow past an obstacle (2D)
We shall be interested in solving the system

\[ \begin{align*}
\rho_z + (\rho u)_x &= 0, \\
u_z + uu_x + \frac{df}{d\rho} \rho_x + \left( \frac{(\rho_x)^2}{8\rho^2} - \frac{\rho_{xx}}{4\rho} \right)_x &= 0,
\end{align*} \]

with initial conditions

\[ \rho(x, 0) = \begin{cases} 
\rho_0 > 1 & \text{for } x < 0, \\
1 & \text{for } x > 0; 
\end{cases} \quad u(x, 0) = 0, \]

that is we assume that initial value \( u(x, 0) \) is equal to zero everywhere, which means that the input beam enters the photo-refractive medium at \( z = 0 \) without any focussing/defocussing.
NLS equation with saturable nonlinearity (sNLS): some general properties

- **A bi-directional equation**: a general initial discontinuity in \( \rho \) and \( u \) resolves into a combination of two waves: DSWs and/or rarefaction waves.

- **The dispersion relation** for the small-amplitude linear waves propagating against a constant background \( \rho = \rho_0, u = u_0 \):

  \[
  \omega = \omega_0(\rho_0, u_0, k) = ku_0 \pm k \sqrt{\frac{\rho_0}{(1 + \gamma \rho_0)^2} + \frac{k^2}{4}}.
  \]

Next we are going to look at:

- **Dispersionless limit**
- **Periodic travelling waves and solitons**
By neglecting dispersion effects for a moment, we arrive at the hydrodynamic-type system

\[
\begin{align*}
\rho_z + (\rho u)_x &= 0, \\
u_z + uu_x + f'(\rho)\rho_x &= 0,
\end{align*}
\]

with

\[
f(\rho) = \frac{\rho}{1 + \gamma \rho}.
\]

System (9) can be easily cast in the Riemann invariant form

\[
\frac{\partial r_\pm}{\partial z} + V_\pm \frac{\partial r_\pm}{\partial x} = 0,
\]

where the Riemann invariants are

\[
r_\pm = u \pm 2 \sqrt{\frac{2}{\gamma}} \arctan \sqrt{\gamma \rho},
\]

and the characteristic velocities are

\[
V_\pm = u \pm \frac{\sqrt{\rho}}{1 + \gamma \rho}.
\]
Periodic solutions and a photo-refractive soliton

We shall look for the solution in the form \( \rho = \rho(\theta), \ u = u(\theta), \) where \( \theta = x - cz \) is the phase and \( c = \text{constant} \) is the phase velocity. Then the sNLS equation reduces to an ODE

\[
\left( \frac{d\rho}{d\theta} \right)^2 = -\frac{8\rho}{\gamma^2} \ln(1 + \gamma \rho) + \left( a_2 + \frac{8}{\gamma} \right) \rho^2 + a_3 \rho + a_4 \equiv Q(\rho),
\]

where \( a_2, \ a_3 \) and \( a_4 \) are arbitrary constants. The function \( Q(\rho) \) has three zeros \( e_1, \ e_2, \ e_3 \) so that \( \rho \) oscillates between \( e_1 \) and \( e_2 \) where \( Q(\rho) \geq 0 \).

In the soliton limit we have the boundary conditions satisfied at infinity:

\[
\rho \to \rho_b, \quad u \to u_b, \quad \frac{d\rho}{d\theta} \to 0, \quad \frac{d^2\rho}{d\theta^2} \to 0 \quad \text{for} \quad |\theta| \to \infty,
\]

plus the condition \( \frac{d\rho}{d\theta} = 0 \) at \( \rho = \rho_m \leq \rho_b \), where \( \rho_m \) is the value of the “density" in the minimum of the dark soliton and \( \rho_b \) is the “background” intensity.
Elementary properties of the sNLS equation

- Dam-break problem
- Simple DSW
- Dark soliton train generation
- Conclusions

Graphs showing the function $Q(\rho)$ for different values of $\gamma$: $\gamma = 0$, $\gamma = 1$, $\gamma = 2$. Graphs also show the dependence of another function on $\theta$.
The soliton velocity $c$ and the inverse soliton half-width $\kappa$ are expressed in terms of the background density $\rho_b$ and the density at the soliton minimum $\rho_m$ as

$$(c - u_b)^2 = \frac{2\rho_m}{\gamma(\rho_b - \rho_m)} \left[ \frac{1}{\gamma(\rho_b - \rho_m)} \ln \frac{1 + \gamma \rho_b}{1 + \gamma \rho_m} - \frac{1}{1 + \gamma \rho_b} \right].$$

Note: the soliton amplitude $a = \rho_b - \rho_m$

$$\kappa = \left[ \frac{8\rho_m + 4\gamma \rho_b(\rho_b + \rho_m)}{\gamma(\rho_b - \rho_m)(1 + \gamma \rho_m)^2} - \frac{8\rho_m}{\gamma^2(\rho_b - \rho_m)^2} \ln \frac{1 + \gamma \rho_b}{1 + \gamma \rho_m} \right]^{1/2}.$$

Above: the dependencies $c(\gamma)$ and $\kappa(\gamma)$ for fixed $\rho_b = 1$, $\rho_m = 0.2$
Modulation equations

The DSW is modeled by the modulated periodic solution of the sNLS equation. The slowly varying parameters of the travelling wave solution (for instance, $\bar{\rho}, \bar{u}, k, a$) satisfy the **Whitham equations**, obtained by averaging of dispersive conservation laws over the periodic family:

$$\frac{\partial}{\partial z} P_j(\bar{\rho}, \bar{u}, k, a) + \frac{\partial}{\partial x} Q_j(\bar{\rho}, \bar{u}, k, a) = 0, \quad j = 1, 2, 3.$$  

$$k_t + (\omega(\bar{\rho}, \bar{u}, k, a))_x = 0.$$  

We assume **hyperbolicity** of the modulation system for the defocusing sNLS, which implies modulational stability.

For $\gamma = 0$ (cubic defocusing NLS); there is a system of modulation variables (**the Riemann invariants**), $(\bar{\rho}, \bar{u}, k, a) \mapsto (r_1, r_2, r_3, r_4)$ such that the modulation system assumes a diagonal form

$$\frac{\partial r_j}{\partial z} + V_j(r) \frac{\partial r_j}{\partial x} = 0, \quad i = 1, \ldots, 4.$$  

which makes a **huge difference** for the analysis.
The NLS extension of the original Gurevich and Pitaevskii (1973) formulation for the KdV modulation system.

Matching conditions at free boundaries $x^\pm(z)$:

Trailing (soliton) edge $x = x^-(z): \quad k = 0, \quad \overline{\rho} = \rho^-, \quad \overline{u} = u^-,$

Leading (harmonic) edge $x = x^+(z): \quad a = 0, \quad \overline{\rho} = \rho^+, \quad \overline{u} = u^+.$

Here $\rho^\pm, u^\pm$ is the solution of the dispersionless limit of the sNLS calculated at the (unknown) boundaries $x^\pm(z)$.

Important: we do not impose any conditions for $k$ at the leading (harmonic) edge and for $a$ at the trailing (soliton) edge.
\( \gamma = 0: \text{cubic NLS equation: some DSW-related results} \)

- Key property of the NLS modulation system: the availability of the Riemann invariants

\[
\frac{\partial r_j}{\partial z} + V_j(r) \frac{\partial r_j}{\partial x} = 0, \quad i = 1, \ldots, 4
\]  

(10)

Then the similarity solutions \( V_3 = x/z, r_{1,2,4} = \text{const}_{1,2,4} \) and \( V_2 = x/z, r_{1,3,4} = \text{const}_{1,3,4} \) describe the modulations in the right- and left- propagating DSWs respectively

– Gurevich and Krylov, JETP 92 (1987) 1684;

- For the focusing case the system (10) remains valid but the Riemann invariants and characteristic velocities form complex conjugate pairs. The similarity modulation solutions describing an initial stage of the “focusing” DSW development was obtained in: El, Gurevich, Khodorovskii and Krylov, Phys. Lett. A (1993) and Kamchatnov, Phys. Rep. (1997)
Major difficulty: absence of the Riemann invariant structure for the modulation system associated with the sNLS equation.

However: it turns out that the key parameters:
- the hydrodynamic transition conditions across the DSW;
- the speeds of the DSW edges;
- the amplitude of the largest soliton
– still can be found exactly even when the Riemann invariants are not available.

Saturable NLS: dam-break problem

Figure: Two-wave resolution in a dam-break problem

First need to find the intermediate state \( \rho^-, u^- \) in terms of \( \rho_0 \). For that we need:

- Rarefaction wave solution
- Transition conditions across the DSW
The rarefaction wave is found as an expansion fan solution of the dispersionless sNLS equation:

\[ r_+ = r_0^+ = \text{constant}, \quad V_-(r_-, r_0^+,:) = x/z \]

or explicitly:

\[ u = \frac{2}{\sqrt{\gamma}} (\arctan \sqrt{\gamma \rho_0} - \arctan \sqrt{\gamma \rho}) , \quad (11) \]

\[ \frac{\sqrt{\rho}}{1 + \gamma \rho} + \frac{2}{\sqrt{\gamma}} (\arctan \sqrt{\gamma \rho} - \arctan \sqrt{\gamma \rho_0}) = -\frac{x}{z} , \quad (12) \]
Simple DSW transition conditions

The transition across the simple right-propagating DSW is asymptotically as $t \to \infty$ characterised by a zero jump for the Riemann invariant $r_-$ of the dispersionless limit equations (as in the simple wave of ‘compression’!) (GE, Chaos 2005). Therefore, for the photorefractive DSW propagating to the right we have the transition condition

$$r_-|_{x-} = r_-|_{x+}.$$

or explicitly

$$u^- = \frac{2}{\sqrt{\gamma}} \left( \arctan \sqrt{\gamma \rho^-} - \arctan \sqrt{\gamma} \right)$$

Note that this does not coincide with the classical shock jump condition. From the rarefaction wave solution we also have

$$u^- = \frac{2}{\sqrt{\gamma}} \left( \arctan \sqrt{\gamma \rho_0} - \arctan \sqrt{\gamma \rho^-} \right)$$

Now, the intermediate state $\rho^-$, $u^-$ is completely determined.
Intermediate state: comparison with numerics

Analytic expressions:

\[ \rho^- = \left[ \frac{\sqrt{1 + \gamma \rho_0} - 1 + \sqrt{\rho_0}(\sqrt{1 + \gamma} - 1)}{\gamma \sqrt{\rho_0} - (\sqrt{1 + \gamma \rho_0} - 1)(\sqrt{1 + \gamma} - 1)} \right]^2, \quad (13) \]

\[ u^- = \frac{2}{\sqrt{\gamma}} \left( \arctan \sqrt{\gamma \rho^-} - \arctan \sqrt{\gamma} \right). \quad (14) \]

**Figure:** Dependence of \( \rho^- \), \( u^- \) on the saturation parameter \( \gamma \) for \( \rho_0 = 5 \). Solid lines: formulae (13), (14). Dots: the values of \( \rho^- \), \( u^- \) obtained from the numerical simulations.
The edges of a DSW

We define the leading \( x^+(z) \) and the trailing \( x^-(z) \) edges of the DSW by the kinematic conditions:

- The leading edge \((a = 0)\) speed is equal to the linear group velocity of the leading wave packet

\[
\frac{dx^+}{dz} = \frac{\partial \omega_0}{\partial k} (\rho^+, u^+, k^+) \equiv s^+, 
\]

Here \( k^+ \) is the wavenumber at the trailing edge.

- The trailing edge \((k = 0)\) speed is equal to the trailing dark soliton velocity

\[
\frac{dx^-}{dz} = c_s (\rho^-, u^-, a^-) \equiv s^- .
\]

Here \( a^- \) is the trailing soliton amplitude and \( c_s (\rho, u, a) \) is the speed-amplitude relationship for a soliton propagating against given background \( \rho, u \).

To find \( s^\pm \) we need to express \( k^+ \) and \( a^- \) in terms of the “density” jump \((\rho^- - \rho^+)\) across the DSW.
The edges of a simple DSW

The values $k^+$ and $a^-$ are found from the condition that the edges of the DSW (the free boundaries $x^\pm(z)$ where the modulation solution is matched with the external “dispersionless” solution) must be the characteristics (more precisely, multiple characteristics) of the modulation system. As a result, one cannot specify all three values $k, \bar{\rho}, \bar{u}$ independently at the leading edge $x^+(z)$ (as well as $a, \bar{\rho}, \bar{u}$ at the trailing edge $x^-(z)$).

Need to find the characteristic forms (ODE’s) for the modulation system in two particular limits: $a = 0$ and $k = 0$. Generally, for that, we would need to know the full system of eigenvalues and eigenvectors for the modulation system and then to pass to the limits as $a \to 0$ and $k \to 0$.

Not a feasible task.

However...
The modulation systems, being obtained by the averaging of conservation laws over the periodic travelling wave solution, must have at least two exact reductions: one for \( a = 0 \) (linear waves) and another one for \( k = 0 \) (solitons). It is clear that in both these limits one has

\[
F(\rho, u) = F(\bar{\rho}, \bar{u})
\]

Then, when \( a = 0 \) or \( k = 0 \) the modulation system must agree with the dispersionless limit of the original sNLS equation.
When $a = 0$ the modulation equations for $\bar{\rho}, \bar{u}, a$ reduces to

$$a = 0, \quad \bar{\rho}_z + (\bar{\rho} \bar{u})_x = 0, \quad \bar{u}_z + \bar{u} \bar{u}_x + f'(\bar{\rho})\bar{\rho}_x = 0. \quad (15)$$

plus

$$k_z + (\omega_0(\bar{\rho}, \bar{u}, k))_x = 0, \quad (16)$$

where

$$\omega_0(\bar{\rho}, \bar{u}, k)) = k \left( \bar{u} + \sqrt{\frac{\bar{\rho}}{(1 + \gamma\bar{\rho})^2 + \frac{k^2}{4}}} \right) \quad (17)$$

is the linear dispersion relation, in which the constant background values $\rho_0$ and $u_0$ are replaced with slowly varying functions $\bar{\rho}(x, z)$ and $\bar{u}(x, z)$, satisfying (15).
\( a = 0: \text{reduced modulation system} \)

To be consistent with the matching condition at \( x = x^+(z) \) and with the simple DSW transition condition we require

\[
\overline{u} = \frac{2}{\sqrt{\gamma}} \left( \arctan \sqrt{\gamma \rho} - \arctan \sqrt{\gamma} \right),
\]

which incidentally is an exact solution of the “hydrodynamic” part of the reduced modulation system for \( a = 0 \)!

Now, substitution of (18) into system (15), (16) further reduces it to only two differential equations

\[
\overline{\rho}_z + V_+(\overline{\rho})\overline{\rho}_x = 0, \quad k_z + (\Omega(\overline{\rho}, k))_x = 0,
\]

where

\[
V_+(\overline{\rho}) = \frac{2}{\sqrt{\gamma}} \left( \arctan \sqrt{\gamma \rho} - \arctan \sqrt{\gamma} \right) + \frac{\sqrt{\rho}}{1 + \gamma \rho},
\]

\[
\Omega(\overline{\rho}, k) = k \left[ \frac{2}{\sqrt{\gamma}} \left( \arctan \sqrt{\gamma \rho} - \arctan \sqrt{\gamma} \right) + \sqrt{\frac{\rho}{(1 + \gamma \rho)^2 + \frac{k^2}{4}}} \right].
\]
a = 0: the characteristics

Two families of characteristics:

\[ \frac{dx}{dz} = V_+(\bar{\rho}) \]  \hspace{1cm} (19)

and

\[ \frac{dx}{dz} = \frac{\partial \Omega(\bar{\rho}, k)}{\partial k} . \]  \hspace{1cm} (20)

- the family (19) transfers “external” hydrodynamic data into the dispersive shock wave region and does not depend on the oscillatory structure (i.e. on \(k\)).
- the family (20) is the linear group velocity family, and by definition the leading edge curve \(x^+(z)\) must belong to this family.

One cannot specify \(k\) and \(\bar{\rho}\) on the same characteristic \(x^+(z)\)!

Therefore, when \(a = 0\) one must have \(k = k(\bar{\rho})\) for the modulation solution we are interested in.
We substitute $k = k(\rho)$ into the system (19) to obtain at once

$$a = 0 : \quad \frac{dk}{d\rho} = \frac{\partial \Omega / \partial \rho}{V_+ - \partial \Omega / \partial k} \quad \text{on} \quad \frac{dx}{dz} = \frac{\partial \Omega}{\partial k}. \quad (21)$$

The above ordinary differential equation for $k$ must be solved with the initial condition $k(\rho^-) = 0$. Indeed, since the equation (21) was derived for the case $a = 0$ it must remain valid in the case of the dispersive shock wave of zero intensity (when $\rho^+ = \rho^-$), so the dependence $k(\rho)$ should correctly reproduce the zero wavenumber condition at the trailing edge where $\rho = \rho^-$. Now, integrating (21) we find $k(\rho)$, and, therefore $k^+ = k(\rho^+)$. Then the leading edge speed is given by

$$s^+ = \frac{\partial \Omega}{\partial k}(\rho^+, k^+)$$
Formula for the leading edge \( s^+ \).

By introducing
\[
\alpha = \sqrt{1 + \frac{k^2(1 + \gamma \rho)^2}{4\rho}}
\]  
(22)

instead of \( k \), the ODE for \( k(\rho) \) reduces to
\[
\frac{d\alpha}{d\rho} = \frac{- (1 + \alpha)[1 + 3\gamma \rho + 2\alpha(1 - \gamma \rho)]}{2\rho(1 + \gamma \rho)(1 + 2\alpha)}.
\]  
(23)

with the initial condition \( \alpha(\rho^-) = 1 \),

The wavenumber \( k^+ \) at the leading edge is then given by
\[
k^+ = k(\rho^+) = \frac{2\sqrt{\alpha^2(\rho^+) - 1}}{1 + \gamma}.
\]  
(24)

Then for the leading edge speed we get (take \( \rho^+ = 1 \))
\[
s^+ = \frac{\partial \Omega}{\partial k}(\rho^+, k^+) = \frac{1}{1 + \gamma} \left( 2\alpha(\rho^+) - \frac{1}{\alpha(\rho^+)} \right).
\]  
(25)

For the case \( \gamma = 0 \) we recover known cubic NLS result (Gurevich & Krylov 1987): \( s^+ = (8\rho^- - 8\sqrt{\rho^-} + 1)/(2\sqrt{\rho^-} - 1) \)
The trailing edge \( x^-(z) \)

The analysis of the trailing (soliton) edge where \( k = 0 \), is more delicate as the limit as \( k \to 0 \) of the wave conservation equation \( k_z + \omega_x = 0 \) is a singular one.

However, the result is completely analogous to that for the leading, zero-amplitude, edge.

The key observation is that the soliton speed can be found as

\[
c_s = \lim_{k \to 0} \frac{\omega}{k} = \frac{\tilde{\omega}_0}{\kappa}
\]

where \( \tilde{\omega}_0(\kappa, \overline{\rho}, \overline{u}) \) is the linear dispersion relation for the conjugate sNLS equation obtained by the change of independent variables \( z \mapsto iz, x \mapsto ix \) i.e.

\[
\tilde{\omega}_0 = -i \omega_0(\overline{\rho}, \overline{u}, i\kappa).
\]

Here \( \kappa \) is the ‘soliton wavenumber’ (inverse half-width) which is related to the sNLS soliton amplitude – we have done it earlier.
\( k = 0: \text{ the characteristic integral} \)

In the soliton limit \( k \to 0 \) the modulation system reduces to

\[
\rho_z + V_+(\rho)\rho_x = 0
\]

(as earlier for \( a \to 0 \)) and

\[
\kappa_z + \tilde{\Omega}_x = 0 \quad \text{on} \quad \frac{dx}{dz} = \frac{\tilde{\Omega}(\rho, \kappa)}{\kappa}
\]

(26)

which are basically the equations for a single dark solitary evolution on a slowly varying simple-wave background. Here

\[
\tilde{\Omega}(\rho, \kappa) = \tilde{\omega}_0(\rho, \bar{u}(\rho), \kappa) = \kappa \left[ \frac{2}{\sqrt{\gamma}} \left( \arctan \sqrt{\gamma \rho} - \arctan \sqrt{\gamma} \right) + \sqrt{\frac{\rho}{(1 + \gamma \rho)^2} - \frac{\kappa^2}{4}} \right].
\]

From the definition of the trailing edge, the characteristic \( x^-(z) \) belongs to the family (26). So one \( \kappa = \kappa(\rho) \) on this characteristic.
\( k = 0 : \text{the characteristic integral} \)

Substituting \( \kappa = \kappa(\rho) \) into the reduced modulation system we obtain

\[
k = 0 : \quad \frac{d\kappa}{d\rho} = \frac{\partial \tilde{\Omega}}{\partial \rho} \quad \text{on} \quad \frac{dx}{dz} = \frac{\tilde{\Omega}(\rho, \kappa)}{\kappa}
\]

(27)

The initial condition for the ODE (27) is \( \kappa(\rho^+) = 0 \) and follows from the requirement that the obtained dependence \( \kappa(\rho) \) should be applicable to the case of the zero-intensity DSW, which corresponds to initial values \( \rho^- = \rho^+ \). In this case, the solitons get infinitely large, that is \( \kappa \rightarrow 0 \) as \( \rho \rightarrow \rho^+ \);
Formulae for the trailing edge speed/amplitude

As a result, we obtain for the trailing edge speed

$$s^- = \frac{2}{\sqrt{\gamma}}(\arctan \sqrt{\gamma \rho^-} - \arctan \sqrt{\gamma}) + \frac{\sqrt{\rho^-}}{1 + \gamma \rho^-} \tilde{\alpha}(\rho^-).$$

where $\tilde{\alpha}(\rho)$ satisfies the ODE

$$\frac{d\tilde{\alpha}}{d\rho} = -\frac{(1 + \tilde{\alpha})[1 + 3\gamma \rho + 2\tilde{\alpha}(1 - \gamma \rho)]}{2\rho(1 + \gamma \rho)(1 + 2\tilde{\alpha})}$$

with the initial condition $\tilde{\alpha}(1) = 1$.

The trailing dark soliton amplitude $a = \rho^- - \rho_m$ is given by

$$\frac{\rho^- \tilde{\alpha}^2(\rho^-)}{(1 + \gamma \rho^-)^2} = \frac{2(\rho^- - a)}{\gamma a} \left[ \frac{1}{\gamma a} \ln \frac{1 + \gamma \rho^-}{1 + \gamma (\rho^- - a)} - \frac{1}{1 + \gamma \rho^-} \right].$$

When $\gamma = 0$ we recover the cubic NLS results by Gurevich and Krylov (1987): $s^- = \sqrt{\rho^-}, a = 4(\sqrt{\rho^-} - 1)$.
Figure: Dependence of the edge speeds $s^\pm$ on the jump $\rho^- - 1$ across for $\gamma = 0.2$.

Figure: Dependence of edge velocities $s^+$ and $s^-$ on the saturation parameter $\gamma$ for fixed values of the intensities at two sides of the dispersive shock: $\rho^- = 2$ and $\rho^+ = 1$.  

DSW edge speeds: plots and comparisons
The trailing soliton amplitude is $a = \rho^- - \rho_m$. It is possible that for some value of the initial step $\rho_{cr}$ we get $a = \rho^-$, which implies that the density (intensity) at the soliton minimum vanishes, $\rho_m = 0$: vacuum point. For the cubic NLS this happens at $\rho^- = 4$. The theoretical dependence of the critical value of $\rho^-$ at which the vacuum point forms at the trailing edge, on the saturation parameter $\gamma$ is shown below.
Large amplitudes: vacuum point

If $\rho^- > \rho_{cr}$ the vacuum point forms inside the DSW:
Dark soliton train generation from a localised pulse

Decaying as $|x| \to \infty$ initial distribution:

$$\rho_0(x) = \left(1 - \frac{1}{\cosh(0.2x)}\right)^2, \quad u_0(x) = \frac{2}{\sqrt{\gamma}}(\arctan \sqrt{\gamma \rho_0(x)} - \arctan \sqrt{\gamma}).$$

Profile of the intensity at $z = 100$ evolved from the initial pulse. With a high accuracy the entire initial disturbance evolves into a dark soliton train (no radiation). No self-similarity $x/z$ anymore.

Can we fix it?
Yes we can! (Details in: El, Grimshaw and Smyth (2008))
Outline:

The “wave conservation” law

\[
\frac{\partial k}{\partial z} + \frac{\partial \omega}{\partial x} = 0
\]

For the decaying profile we have \( \omega \to 0 \) as \(|x| \to \infty\).

Then the total “number of waves” is conserved and can be calculated as

\[
N \approx \frac{1}{2\pi} \int_{-\infty}^{+\infty} k \, dx = \text{constant.} \tag{28}
\]

and it does not depend on \( z \)!

So (28) gives a total number of solitons as \( z \gg 1 \).

On the other hand, if we knew the distribution \( k(x, 0) \) we could compute \( N \) for \( z = 0 \), which is the same.
The analysis is again based on the characteristic integrals.

**The idea:**

- We know the dependence $k(\rho)$ along the leading (harmonic) edge of the DSW.
- Outside the DSW we have the simple wave $\rho_z + V_+ (\rho) \rho_x = 0$.
- We “project” the characteristic integral $k(\rho)$ along the simple-wave family $dx/dz = V_+ (\rho)$ onto the $x$-axis to find $k(x,0)$ for $x > 0$.
- For $x < 0$ (the soliton edge side) we simply have $k(x,0) = 0$. 
As a result we obtain for the total number of solitons in the train

\[ N \approx \frac{1}{2\pi} \int_{-\infty}^{+\infty} k(x, 0)dx = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sqrt{\rho_0(x)(\alpha_0^2(x) - 1)}}{1 + \gamma \rho_0(x)} dx, \]

where \( \alpha_0(x) = \alpha(\rho_0(x)) \) and for \( \alpha(\rho) \) we have the (characteristic integral) ODE

\[ \frac{d\alpha}{d\rho} = -\frac{(1 + \alpha)[1 + 3\gamma \bar{\rho} + 2\alpha(1 - \gamma \bar{\rho})]}{2\bar{\rho}(1 + \gamma \bar{\rho})(1 + 2\alpha)}. \]

with the initial condition \( \alpha(1) = 1 \).

For the cubic NLS case \( \gamma = 0 \) we recover the Bohr-Sommerfeld type (semi-classical quantization) result for the total number of bound states (Jin, Levermore, McLaughlin, CPAM 1999; Kamchatnov et.al PRE 2002)

\[ N \approx \frac{2}{\pi} \int_{-\infty}^{+\infty} \sqrt{1 - \rho_0^{1/2}} dx \]
Dark soliton train generation: comparison with numerical simulations

Number of dark solitons $N$ as a function of $\gamma$; solid line: analytical curve; dots: numerical simulations.
Conclusion:
- It is possible to effectively study the dynamics of dispersive shock waves in the frameworks of non-integrable conservative models.

Perspectives:
- DSW in nonlocal media (nematic crystals, in particular)
- Circular DSW
- DSW in focusing media
Thank you