Nonlinear Waves

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In many physical situations it is necessary to take account of the fact that nonlinear waves propagate through a variable environment. Examples include shallow-water waves moving over an uneven bottom, internal gravity waves in lakes of varying cross-section, long waves in rotating fluids contained in cylindrical tubes, and many others. Typically, the problems of this type imply that the coefficients in the governing nonlinear dispersive wave equation are varying in time and/or space. Here we consider some problems modeled by the \textit{variable-coefficient KdV} equation
\begin{equation}
 u_t + \alpha(t) uu_x + \beta(t) u_{xxx} = 0 ,
\end{equation}

- In the modelling of the variable environment effects, the variables $x, t$ in equation (1) are not necessarily the same physical space and time coordinates as in the traditional interpretation of the KdV equation.
- Generally, equation (1) is not integrable by the IST with some exceptions, when the coefficients $\alpha(t)$ and $\beta(t)$ satisfy certain relations.
Assuming $\alpha \neq 0$ we introduce the new variables

$$t' = \frac{1}{6} \int_0^t \alpha(\hat{t}) d\hat{t}, \quad \lambda(t') = \frac{6\beta}{\alpha},$$

so that, on omitting the superscript for $t$, equation (1) becomes

$$u_t + 6uu_x + \lambda(t) u_{xxx} = 0.$$  \hspace{1cm} (3)

Another form of the perturbed KdV equation arises when one considers effects of dissipation (or energy input) modelled by the the right-hand side term

$$u_t + 6uu_x + u_{xxx} = R(u, u_x, u_{xx}...).$$  \hspace{1cm} (4)

Generally equations (3), (4) should be treated numerically, however, significant analytical progress is possible if (3) or (4) represents a weakly perturbed KdV equation, i.e. when $\lambda(t) \equiv \lambda(\epsilon t), \quad \epsilon \ll 1$ (slowly varying coefficients) or $R(u, u_x....) \equiv \epsilon \tilde{R}$ (weak dissipation).
11.2 Propagation of the KdV solitary wave through an inhomogeneous conservative medium (e.g. shallow-water solitary wave over uneven frictionless bottom)

Generally: numerical solution. Analytical progress is possible in two contrasting cases.

Case 1: **Fast depth variation**

Case 2: **Slow depth variation**
11.2a Case 1. Fast depth change: soliton fission

Propagation of a shallow-water solitary wave over a sloping bottom is modelled by the variable-coefficient KdV equation

\[ u_t + 6uu_x + \lambda(t)u_{xxx} = 0, \tag{5} \]

Note: Independent variables \( x \) and \( t \) in (5) are generally not physical space and time.

Case 1: Solitary wave advancing into a rapidly decreasing depth region. This is modelled by the following behaviour of \( \lambda(t) \) in (5):
\( \lambda = \lambda_1 \) for \( t < 0; \) \( \lambda = \lambda_2 \) for \( t > 0. \)

Result: soliton fission
Soliton fission: Mathematical formulation

For $t < 0$:

$$u(x, t) = a \, \text{sech}^2\left[\sqrt{a/(2\lambda_1)}(x - c_s t)\right], \quad c_s = 2a$$

which is the solitary wave solution of the KdV equation

$$u_t + 6uu_x + \lambda u_{xxx} = 0$$

with $\lambda = \lambda_1 > 0$.

At $t = 0$ we have $u(x, 0) = a \, \text{sech}^2\left[\sqrt{a/(2\lambda_1)}x\right]$, which is then considered as the initial condition for the KdV equation (7) with $\lambda = \lambda_2$.

Note: For $t > 0$, function (6) is no longer a solution of the governing KdV equation (7) so one uses the IST to find the solution. The outcome depends on the sign of $\lambda_2$

- $\lambda_2 > 0$: At $t \gg 1$ one has $N$ solitons (discrete spectrum)
  $$u_n(x, 0) = a_n \, \text{sech}^2\left[\sqrt{a_n/(2\lambda_2)}(x - a_n t)\right], \quad n = 1, 2, \ldots, N$$
  and some linear radiation (continuous spectrum). The amplitudes $a_n$ and the number $N$ of the solitons, as well as the parameters of the radiation, are found from the IST.

- $\lambda_2 < 0$: The initial soliton (6) completely transforms into a linear radiation as $t \to \infty$. 
11.2a Case 2. Slow depth change: adiabatic variation of the solitary wave

If the depth $h$ changes slowly with physical coordinate $x$, one has for the coefficient $\lambda$ in the corresponding variable coefficient KdV equation (5):

$$\lambda = \lambda(T), \quad T = \epsilon t, \quad \epsilon \ll 1.$$  (8)

Here $\epsilon$ measures the slope. Then, the solution of the KdV equation

$$u_t + 6uu_x + \lambda(T)u_{xxx} = 0$$  (9)

can be sought in the form of the asymptotic expansion

$$u = u_0 + \epsilon u_1 + \ldots,$$  (10)

where the leading term is given by the slowly varying solitary wave solution

$$u_0 = a \sech^2 \left\{ \gamma(x - \frac{\Phi(T)}{\epsilon}) \right\},$$  (11)

so that

$$d\Phi/dT = c_s = 2a = 4\lambda \gamma^2.$$  (12)

Note that for $\lambda = \text{constant}$ formulae (11), (12) reduce to the standard KdV soliton expression $u(x, t) = a \sech^2 \left[ \sqrt{a/(2\lambda)}(x - c_s t) \right], c_s = 2a$.  

The variations of the amplitude $a$ and the inverse half-width parameter $\gamma$ with the slow time variable $T$ are determined by noticing that the variable-coefficient KdV equation (3) possesses the momentum conservation law

$$\int_{-\infty}^{\infty} u^2 dx = \text{constant}. \quad (13)$$

Substitution of (11) into (13) readily shows that

$$\frac{\gamma}{\gamma_0} = \left( \frac{\lambda_0}{\lambda} \right)^{2/3}, \quad (14)$$

where the subscript ‘0’ indicates quantities evaluated at some fixed moment $T = T_0$, say for $\lambda = \lambda_0$.

It follows from (11), (12) and (14) that the slowly-varying solitary wave, $u_0$ is now completely determined.
The outcome: adiabatically varying solitary wave. Generally:

- For a wave advancing into decreasing depth, there is a tendency to increase the amplitude.

- For a wave advancing into increasing depth, there is a tendency to decrease the amplitude.

The detailed amplitude variations are determined by the slowly changing bottom profile $h(x)$ which is translated into the function $\lambda(T)$. 
11.3. Formation of a trailing shelf

So, the slowly-varying solitary wave is completely determined by the leading term of the asymptotic expansion (10) $u = u_0 + \epsilon u_1 + \ldots$. However, the variable-coefficient KdV equation (3) also has a conservation law for the ‘mass’

$$\int_{-\infty}^{\infty} u \, dx = \text{constant},$$

which is not satisfied by $u_0(x, t)$ (check!!!). The situation can be remedied by taking into account the next term in the asymptotic expansion (10) and allowing $\int_{-\infty}^{\infty} u_1(x) \, dx$ be $O(\epsilon^{-1})$.

More precisely, conservation of mass is assured by the generation of a trailing shelf $u_s$, such that $u = u_0 + u_s$ where $u_s$ typically has an amplitude $O(\epsilon)$ and is supported on the interval $0 < x < \Phi(T)/\epsilon$. Thus, the shelf stretches over a zone of $O(\epsilon^{-1})$, and hence carries $O(1)$ mass.
To find $u_s$, we substitute $u = u_0 + u_s$ into the conservation of mass equation (15) to obtain
\[
\int_{-\infty}^{\Phi(T)/\epsilon} u_s \, dx + \int_{-\infty}^{\infty} u_0 \, dx = \text{constant},
\]
which, together with the KdV equation $u_t + 6uu_x + \lambda(T)u_{xxx} = 0$ completely determines the dynamics of the shelf $u_s(x, t)$.

Remarks

- The trailing shelf can have positive or negative polarity depending on the behaviour of the function $\lambda(T)$. Namely, the polarity of the shelf $\sigma = -\text{sgn}(\lambda_T)$.

- If $\sigma > 0$ (elevation) then, according to the IST, the shelf will decompose as $t \to \infty$ into a large number of small-amplitude solitons, which is consistent with the fission scenario for rapidly varying environment. If $\sigma < 0$ (depression) then the shelf asymptotically transforms into a linear dispersing wave packet.
11.3 Evolution of the trailing shelf of elevation.

\begin{align*}
\text{\textit{t} = 0} & \\
\text{\textit{t} \propto \varepsilon^{-1}} & \\
\text{\textit{t} \propto \varepsilon^{-2}} & \\
\end{align*}
11.4 Effect of weak dissipation.

Effects of a weak dissipation (damping) are modelled by the perturbed KdV equation

$$u_t + 6uu_x + u_{xxx} = R(u), \quad (17)$$

where the function $R(u)$ can have different forms depending on the specific dissipation mechanism

- **Linear damping** $R(u) = -\mu u$,  
- **Volume viscosity** $R(u) = \mu u_{xx}$,  
- **Chezy (turbulent) friction** $R(u) = -\mu u|u|$  
- **Boundary-layer (nonlocal) damping**
  \[ R(u) = -\mu \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\xi)^{1/2} \exp(i\xi x) \mathcal{F}(u) d\xi, \text{ where} \]
  \[ \mathcal{F}(u) = \int_{-\infty}^{\infty} \exp(-i\xi x) u(x) dx \]

Here $0 < \mu \ll 1$ is the small dissipation coefficient.

**Overall effect of dissipation would be decreasing of the wave amplitude.** Analytic approach to find the slowly varying, due to weak dissipation, solitary wave is analogous to that developed for the variable-coefficient KdV equation (multiple-scale expansions).
Example: Linear damping

The variable-coefficient KdV equation with linear damping in a general form:

$$u_t + 6\alpha(t)uu_x + \beta(t)u_{xxx} = -(f_t/f)u,$$

(18)

where $f(t)$ is some positive decreasing function; $|f_t/f| \ll 1$. We introduce $\tilde{u} = fu$, $\tilde{\alpha} = \alpha/f$. Then for $\tilde{u}$ we obtain the homogeneous variable-coefficient KdV equation

$$\tilde{u}_t + 6\tilde{\alpha}(t)\tilde{u}\tilde{u}_x + \beta(t)\tilde{u}_{xxx} = 0$$

(19)

which we have already considered.
Fluid dynamics application: shallow-water solitary wave on a slope with bottom friction

The relevant variable-coefficient perturbed KdV equation has the form in dimensional variables

\[ A_t + cA_x + \frac{c_x}{2} A + \frac{3c}{2h} AA_x + \frac{ch^2}{6} A_{xxx} = -C_D \frac{c}{h^2} |A| A. \quad (20) \]

Here \( A(x, t) \) is the free surface elevation above the undisturbed depth \( h(x) \), \( c(x) = \sqrt{gh(x)} \) is the linear long wave phase speed and \( C_D \) is a non-dimensional drag coefficient.

Colour code:
- Red: dominant term: translation with the speed \( c \)
- Green: depth variations (slope)
- Brown: nonlinearity
- Blue: dispersion
- Black: turbulent bottom friction (Chezy law)

Balance of terms in (20):

\[ A \sim h^3 \frac{\partial^2}{\partial x^2}, \quad \frac{c_x}{c} \sim h^2 \frac{\partial^3}{\partial x^3}, \quad C_D \sim h \frac{\partial}{\partial x} \]
Within this balance of terms equation (20) can be asymptotically reduced to the constant-coefficient perturbed KdV equation

\[ U_T + 6UU_X + U_{XXX} = F(T)U - G(T)|U|U \tag{21} \]

where \( F(T) = -\frac{9h_T}{4h}, \quad G(T) = 4C_D \frac{g^{1/2}}{h^{1/2}}. \tag{22} \)

Here the variables \( U, X, T \) are connected with the original \( A, x, t \) by the relations

\[ U = \frac{3g}{2h^2} A, \quad T = \frac{1}{6g^{3/2}} \int_0^x \sqrt{h(x)} \, dx, \quad X = \int_0^x \frac{dx'}{c(x')} - t \tag{23} \]
Shallow-water solitary wave on a slope with bottom friction

Next, assuming that $h_T \ll 1$, $C_D \ll 1$ the slowly varying solitary wave solution is given by, at leading order,

$$U(X, T) = 2\gamma^2 \text{sech}^2[\gamma(X - \Phi(T))] ,$$  \hspace{0.5cm} (24)

where \[ \frac{d\Phi}{dT} = c_s = 4\gamma^2 . \hspace{0.5cm} (25) \]

Now, we substitute (24), (25) into the “momentum” balance equation derived from the perturbed KdV equation (21)

$$P_T = 2F(T)P - G(T)Q ,$$  \hspace{0.5cm} (26)

where \[ P = \int_{-\infty}^{+\infty} \frac{U^2}{2} dX , \quad Q = \int_{-\infty}^{\infty} |U|^3 dX , \hspace{0.5cm} (27) \]

and obtain the ordinary differential equation for $\gamma(T)$,

$$\frac{\partial \gamma}{\partial T} = \frac{2}{3} F(T)\gamma - \frac{16}{15} G(T)\gamma^3 .$$  \hspace{0.5cm} (28)

which can be easily integrated to give slow variations with $T$ (and therefore with $x$ by (23)) of the solitary wave inverse width $\gamma$ and, consequently, of the amplitude $a = 2\gamma^2$. 
As a result, we obtain an explicit formula for the solitary wave amplitude variations with $x$ (Miles 1983; El, Grimshaw & Kamchatnov 2007)

$$a = a_0 \left( \frac{h_0}{h} \right) \left[ 1 + \frac{16}{15} C_D a_0 h_0 \int_0^x \frac{dx}{h^3} \right]^{-1},$$

where $a_0$ is the solitary wave amplitude at $x = 0$, where $h = h_0$. For $C_D = 0$ this reduces to the classical Boussinesq (1872) result $a \sim h^{-1}$, while for $h = h_0$ it reduces to the well-known algebraic decay law $a \sim 1/(1 + \text{constant } x)$ due to Chezy friction.


