Nonlinear Waves

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Lecture 18: The Whitham Modulation Theory - II
Our main example in this lecture is the Whitham system for the KdV equation,

\[ u_t + 6uu_x + u_{xxx} = 0. \]  

Three ingredients of the derivation of the Whitham equations:

- **Periodic travelling wave solution** \( u = u(x - ct) \) characterised by three parameters (constants of integration) \( b_1, b_2, b_3 \) (see Appendix):
  \[
  \frac{(u_\theta)^2}{2} = (u - b_1)(u - b_2)(b_3 - u),
  \]

  The phase velocity \( c \) and the wave period \( L \) are

  \[
  c = 2(b_1 + b_2 + b_3), \quad L = \int_0^L d\theta = \sqrt{2} \int_{b_2}^{b_3} \frac{du}{\sqrt{(u - b_1)(u - b_2)(b_3 - u)}}.
  \]

- **Three conservation laws** (after the averaging, the third conservation law can be replaced by the “wave conservation” equation \( k_T + \omega_X = 0 \), so we actually need only two)
  \[
  u_t + (3u^2 + u_{xx})_x = 0; \quad (\frac{1}{2}u^2)_t + (2u^3 + uu_{xx} - \frac{1}{2}u_x^2)_x = 0
  \]
The Whitham system for the KdV equation

- Averaging over the period of the travelling wave solution

\[ \overline{F}(b_1, b_2, b_3) = \frac{1}{L} \int_0^L F(u(\theta; \mathbf{b})) d\theta = \frac{\sqrt{2}}{L} \int_{b_2}^{b_3} \frac{F(u) du}{\sqrt{(b_3 - u)(u - b_2)(u - b_1)}}. \]

Now we introduce slow variables \( X = \varepsilon x, \ T = \varepsilon t, \ \varepsilon \ll 1 \) and allow \( b_j = b_j(X, T), \ j = 1, 2, 3. \)

Then the modulation system for \( b_j(X, T) \) is obtained by averaging of the two KdV conservation laws over the period of the travelling wave solution,

\[ (\overline{u})_T + (3\overline{u}^2 + u_{\theta\theta})_X = 0, \ \ (\frac{1}{2}\overline{u}^2)_T + (2\overline{u}^3 + uu_{\theta\theta} - \frac{1}{2}u_{\theta}^2)_X = 0 \quad (3) \]

and closing (3) with the ”wave conservation” equation

\[ k_T + \omega_X = 0. \quad (4) \]

where \( k(b_1, b_2, b_3) = \frac{2\pi}{L}, \ \omega(b_1, b_2, b_3) = kc. \)

Equations (3), (4) comprise a closed hydrodynamic type system for \( b_1(X, T), b_2(X, T), b_3(X, T) – \text{the KdV -Whitham system.} \)
It was discovered by Whitham (1965) that, upon introducing symmetric combinations

\[ r_1 = \frac{b_1 + b_2}{2}, \quad r_2 = \frac{b_1 + b_3}{2}, \quad r_3 = \frac{b_2 + b_3}{2}, \quad (5) \]

the KdV-Whitham system assumes the diagonal (Riemann) form

\[ \frac{\partial r_j}{\partial T} + V_j(r_1, r_2, r_3) \frac{\partial r_j}{\partial X} = 0, \quad j = 1, 2, 3, \quad (6) \]

where no summation over the repeated indices is assumed and the characteristic velocities have the form

\[ V_1 = 2(r_1 + r_2 + r_3) - 4(r_2 - r_1) \frac{K(m)}{K(m) - E(m)}, \]

\[ V_2 = 2(r_1 + r_2 + r_3) - 4(r_2 - r_1) \frac{(1 - m)K(m)}{E(m) - (1 - m)K(m)}, \]

\[ V_3 = 2(r_1 + r_2 + r_3) + 4(r_3 - r_1) \frac{(1 - m)K(m)}{E(m)}. \]

Here \( K(m) \) and \( E(m) \) are the complete elliptic integrals of the first and the second kind respectively.

The variables \( r_j \) are the Riemann invariants.
The structure of the characteristic velocities

Consider the wave conservation law (4)

\[ k_T + \omega_X = 0 , \quad (*) \]

where now \( k = k(r) \), \( \omega = \omega(r) \). Introduce the Riemann invariants in (*) explicitly:

\[ \sum_{i=1}^{3} \left\{ \frac{\partial k}{\partial r_j} \frac{\partial r_j}{\partial T} + \frac{\partial \omega}{\partial r_j} \frac{\partial r_j}{\partial X} \right\} = 0 , \quad (**) \]

We know that equation (\(*)(*)\), being part of the modulation system, must be consistent with its Riemann form (6)

\[ \frac{\partial r_j}{\partial T} + V_j(r_1, r_2, r_3) \frac{\partial r_j}{\partial X} = 0 , \quad j = 1, 2, 3 . \quad (***) \]

Now substituting (***\emph{\textsuperscript{**}}\textsuperscript{**}) into (**) yields

\[ \sum_{j=1}^{3} \left\{ \frac{\partial \omega}{\partial r_j} - V_j \frac{\partial k}{\partial r_j} \right\} \frac{\partial r_j}{\partial X} = 0 \quad \Rightarrow \quad V_i = \frac{\partial \omega / \partial r_j}{\partial k / \partial r_j} , \quad i = 1, 2, 3 . \]

(the derivatives \( \partial r_j / \partial X \) are independent). So we have arrived at a nonlinear generalization of the linear group velocity notion \( \partial \omega_0 / \partial k \).
The structure of the characteristic velocities

Using \( k = 2\pi/L \) and \( \omega = kc \) we get

\[
V_i = \frac{\partial_i \omega}{\partial_i k} = (1 - \frac{L}{\partial_i L} \partial_i) c, \quad \partial_i \equiv \frac{\partial}{\partial r_i}
\]  

(7)

Now expressing \( L \) and \( c \) in terms of \( r_j \) explicitly we get

\[
L = \int_{r_1}^{r_2} \frac{d\lambda}{\sqrt{(\lambda - r_1)(r_2 - \lambda)(r_3 - \lambda)}} \quad c = 2(r_1 + r_2 + r_3). 
\]

(8)

Calculating \( L \) in terms of the complete elliptic integral of the first kind \( K(m) \) we get

\[
L = \frac{2K(m)}{(r_3 - r_1)^{1/2}}, \quad m = \frac{r_2 - r_1}{r_3 - r_1} 
\]

(9)

Substituting (9) into (7) we obtain explicit formulae for the characteristic velocities \( V_j(r) \) in terms of the complete elliptic integrals.

We shall call the representation (7) for the characteristic velocities a “potential” representation, the phase velocity \( c \) being the potential.

It is instructive to study the behaviour of the Whitham system in two important limiting cases: \( m = 0 \) (harmonic limit) and \( m = 1 \) (soliton limit)
The structure of the Whitham equations: two limits

- **Harmonic limit,** \( m \to 0 \)
  Direct calculation shows: \( V_3 \to 6r_3; \ V_2 \to V_1 \to (12r_1 - 6r_3) \) so the Whitham system reduces to

\[
\frac{\partial r_3}{\partial T} + 6r_3 \frac{\partial r_3}{\partial X} = 0, \quad \frac{\partial r_1}{\partial T} + (12r_1 - 6r_3) \frac{\partial r_1}{\partial X} = 0. \tag{10}
\]

- **Soliton limit,** \( m \to 1 \)
  Now we have: \( V_2 \to V_3 \to (2r_1 + 4r_3); \ V_1 \to 6r_1 \) so that the Whitham system reduces to

\[
r_2 = r_3, \quad \frac{\partial r_1}{\partial T} + 6r_1 \frac{\partial r_1}{\partial X} = 0, \quad \frac{\partial r_3}{\partial T} + (2r_1 + 4r_3) \frac{\partial r_3}{\partial X} = 0. \tag{11}
\]

Thus, in both limits, one of the Whitham equations converts into the Hopf equation \( r_T + 6rr_X = 0 \) (the dispersionless limit of the KdV equation). Using explicit limiting expressions for \( k \) and \( c \) one can show that the multiple characteristic velocity in (10) \( V_2 = V_1 \) coincides with the **linear group velocity** \( \frac{\partial \omega_0}{\partial k} \) while the multiple characteristic velocity \( V_2 = V_3 \) in (11) coincides with the **soliton velocity**, the soliton amplitude being \( 2(r_3 - r_1) \). This is left as an exercise.
Analysis of the characteristic velocities $V_j$ shows that

- the characteristic velocities are real and distinct so the KdV-Whitham system is hyperbolic. Hence:
  1. modulational stability.
  2. classical theory of characteristics for the hydrodynamic type systems is applicable.

- $\frac{\partial V_i}{\partial r_i} > 0$, therefore the KdV-Whitham system is genuinely nonlinear. Thus one can expect the wave breaking effects and complications with the existence of global solution.
Integrability of the KdV-Whitham equations:

1. Simple wave solutions

First, one can observe that the Riemann form
\[ \frac{\partial r_j}{\partial T} + V_j(r_1, r_2, r_3) \frac{\partial r_j}{\partial X} = 0, \quad j = 1, 2, 3, \]
implies that any \( r_j = \text{constant} \) is an exact solution of the Whitham equations. This enables one to consider exact reductions of the Whitham system and find some important particular solutions.

**Simple wave reduction.**
We consider the reduction of the Whitham system when two of the Riemann invariants, say \( r_2 \) and \( r_3 \), are constant, \( r_2 = r_{20}, \quad r_3 = r_{30} \). Then the Whitham system reduces to a single “simple-wave” equation
\[ \frac{\partial r_1}{\partial T} + V_1(r_1, r_{20}, r_{30}) \frac{\partial r_1}{\partial X} = 0 \]
which is integrated using characteristics to give an exact solution
\[ X - V_1(r_1, r_{20}, r_{30}) T = W(r_1) \quad (12) \]
Here \( W(r_1) \) is an arbitrary function. In the solution of an initial-value problem \( W(r_1) \) has the meaning of the inverse to the initial function \( r_1(X, 0) \).
Integrability of the KdV-Whitham equations:

2. Hodograph Solutions

Let \( r_3 = r_{30} = \text{constant} \). Then for the remaining two \( r_{1,2}(X, T) \), one has a \( 2 \times 2 \) system, which can be solved (linearised) using the classical hodograph transformation provided \( r_{1X} \neq 0, r_{2X} \neq 0 \).

This is achieved through the “swap” of the dependent and independent variables \( (r_1, r_2) \leftrightarrow (X, T) \). For that, we write the differentials

\[
dr_1 = \frac{\partial r_1}{\partial T} dT + \frac{\partial r_1}{\partial X} dX
\]

\[
dr_2 = \frac{\partial r_2}{\partial T} dT + \frac{\partial r_2}{\partial X} dX
\]

and solve this system for \( dX \) and \( dT \) to obtain the derivatives

\[
\frac{\partial X}{\partial r_1} = \frac{1}{\Delta} \frac{\partial r_2}{\partial T}, \quad \frac{\partial X}{\partial r_2} = -\frac{1}{\Delta} \frac{\partial r_1}{\partial T}, \quad \frac{\partial T}{\partial r_1} = -\frac{1}{\Delta} \frac{\partial r_2}{\partial X}, \quad \frac{\partial T}{\partial r_2} = \frac{1}{\Delta} \frac{\partial r_1}{\partial T}
\]

where

\[
\Delta = \left\{ \frac{\partial r_1}{\partial X} \frac{\partial r_2}{\partial T} - \frac{\partial r_1}{\partial T} \frac{\partial r_2}{\partial X} \right\} \neq 0
\]

is the Jacobian of the transformation \( (r_1, r_2) \leftrightarrow (X, T) \).
Now, substituting the expressions for $\partial_{T} r_{1,2}$ and $\partial_{X} r_{1,2}$ into the Whitham system we obtain a system of two linear equations for $X(r_{1}, r_{2}), T(r_{1}, r_{2})$:

$$\partial_{1} X - V_{2}(r_{1}, r_{2}, r_{30}) \partial_{1} T = 0, \quad \partial_{2} X - V_{1}(r_{1}, r_{2}, r_{30}) \partial_{2} T = 0,$$

where $\partial_{j} \equiv \partial / \partial r_{j}$. Next, we introduce in (*) substitutions

$$W_{1}(r_{1}, r_{2}) = X - V_{1} T, \quad W_{2}(r_{1}, r_{2}) = X - V_{2} T,$$

to cast it in the form of a symmetric system for $W_{1,2}$:

$$\frac{\partial_{1} W_{2}}{W_{1} - W_{2}} = \frac{\partial_{1} V_{2}}{V_{1} - V_{2}}, \quad \frac{\partial_{2} W_{1}}{W_{2} - W_{1}} = \frac{\partial_{2} V_{1}}{V_{2} - V_{1}}.$$

Now, any solution of the linear system (***') will generate, via (**), a local smooth solution $\{r_{1}(X, T), r_{2}(X, T), r_{30}\}$ of the Whitham system. One can see that analogous systems can be obtained for any two pairs of the Riemann invariants, provided the third invariant is constant. Note:

- the requirement $\Delta \neq 0$ is essential
- the hodograph solution (**) has the form similar to the simple wave solution (12). The crucial difference is that the functions $W_{1,2}$ in (**') are not arbitrary.
Integrability of the KdV-Whitham equations:  
3. Generalised Hodograph Transform

In 1985 Tsarev showed that even if all three Riemann invariants vary, any smooth non-constant solution of the Whitham system (6) can be obtained from the algebraic system

\[ X - V_j(r_1, r_2, r_3) T = W_j(r_1, r_2, r_3), \quad i = 1, 2, 3, \quad (13) \]

where the functions \( W_j \) are found from the overdetermined system of linear partial differential equations,

\[ \frac{\partial_i W_j}{W_i - W_j} = \frac{\partial_i V_j}{V_i - V_j}, \quad i, j = 1, 2, 3, \quad i \neq j. \quad (14) \]

Now, the condition of integrability of the nonlinear diagonal system (6) is reduced to the condition of consistency for the overdetermined linear system (14), which has the form (see Lecture 16)

\[ \partial_i \left( \frac{\partial_j V_k}{V_j - V_k} \right) = \partial_j \left( \frac{\partial_i V_k}{V_i - V_k} \right), \quad i \neq j, \ i \neq k, \ j \neq k. \quad (15) \]

It is not difficult to show that the characteristic velocities (7) satisfy (15) so the KdV-Whitham system (6) is integrable via the generalised hodograph transform (13), (14).
There is a deep connection between the Whitham equations (6) and the spectral problem associated with the original KdV equation. This connection was discovered and thoroughly studied in the paper by Flaschka, Forest and McLaughlin (FFM)(1979). Let us consider the cnoidal wave solution

\[ u_{cn}(x, t) = b_2 + (b_3 - b_2) \text{cn}^2 \left( \sqrt{2(b_3 - b_1)}(x - ct - x_0); m \right) \]  

(16)

taken with negative sign, as a potential in the linear Schrödinger equation in the associated spectral problem,

\[ (-\partial_{xx}^2 - u_{cn}) \phi = \lambda \phi \]

It is well known that the spectrum of the periodic Schrödinger operator generally consists of an infinite number of disjoint intervals called bands. Correspondingly, the ‘forbidden’ zones between the bands are called gaps.
The unique property of the cnoidal wave solution (26) is that its spectrum contains only one finite band. To be exact, the spectral set for the potential $-u_{cn}(x)$ is $S = \{ \lambda : \lambda \in [\lambda_1, \lambda_2] \cup [\lambda_3, \infty) \}$.

This fact had been known long before the creation of the soliton theory in connection with the so-called Lamè potentials. The soliton studies showed that the cnoidal wave solutions of the KdV equation represent the simplest case of potentials belonging to a general class of the so-called finite-gap potentials discovered by Novikov (1974) and Lax (1975). These finite-gap potentials can be expressed in terms of Riemann theta-functions and give rise to multiphase almost periodic solutions of the KdV equation.
It is clear that the cnoidal wave solution can be parametrised by three spectral parameters $\lambda_1, \lambda_2, \lambda_3$ instead of the roots of the polynomial $b_1, b_2$ and $b_3$ (see equation (5)). The remarkable general fact established by FFM is that the Riemann invariants of the Whitham system (6) coincide with the endpoints of the spectral bands of finite-gap potential. In particular, for the single-gap solution (the cnoidal wave), $r_1 = \lambda_1$, $r_2 = \lambda_2$, $r_3 = \lambda_3$. Thus, the spectral problem provides one with the most convenient set of modulation parameters (the Riemann invariants) and, therefore, the Whitham equations (6) describe slow evolution of the spectrum of periodic (generally – quasiperiodic) KdV solutions.

The general theory of finite-gap integration and the spectral theory of the Whitham equations are quite technical. However, in the case of the single-phase waves, which is the most important from the viewpoint of fluid dynamics applications, a simple universal method has been developed by Kamchatnov (2000), enabling one to construct periodic solutions and the Whitham equations directly in Riemann invariants for a broad class of integrable nonlinear dispersive wave equations.
Integrability of the KdV-Whitham equations:
5. Commuting hydrodynamic flows (symmetries)

We have shown that integrability of the KdV equation is inherited by the averaged (Whitham) system. Integrability of the KdV equation is based on the Lax pair $L, A$ and implies existence of an infinite number of commuting flows – “higher” KdV equations generated by the operators commuting with $L$. One can expect that the KdV-Whitham system will also have an infinite number of commuting flows (symmetries).

Let us consider a hydrodynamic type system

$$\frac{\partial r_j}{\partial \tau} + W_j(r_1, r_2, r_3) \frac{\partial r_j}{\partial X} = 0, \quad j = 1, 2, 3,$$  \hspace{1cm} (17)

where $\tau$ is a new “time” variable and $W_j(r)$ are some characteristic velocities. Now we require that system (17) commutes with the KdV-Whitham system (6), i.e. we impose the condition

$$\partial_\tau \partial_\tau r_j = \partial_\tau \partial_\tau r_j.$$

Then (18) leads to a number of constraints on functions $W_i(r)$, which turn out to coincide with the generalised hodograph equations (14). So the generalised hodograph equations (14) are also equations for the hydrodynamic symmetries of the KdV-Whitham system.
Appendix. KdV equation: periodic travelling wave solutions

We shall describe the Whitham method using the KdV equation as the main example. We take it in the canonical form

$$ u_t + 6uu_x + u_{xxx} = 0. \quad (19) $$

Since our aim is to describe slow evolution of nonlinear periodic waves, we start with looking for a solution of (19) in the form of a single-phase travelling wave of a permanent shape, i.e. in the form $u(x, t) = u(\theta)$, where $\theta = x - ct$ is the phase and $c = \text{constant}$ is the phase velocity. For such solutions, the KdV equation reduces to the ordinary differential equation

$$ -cu_\theta + 6uu_\theta + u_{\theta\theta\theta} = 0, \quad (20) $$

which is integrated twice to give

$$ (u_\theta)^2/2 = (u - b_1)(u - b_2)(b_3 - u) \equiv G(u), \quad (21) $$

where $b_3 \geq b_2 \geq b_1$ are constants and

$$ c = 2(b_1 + b_2 + b_3). \quad (22) $$
Equation (21), \( (u_\theta)^2/2 = (u - b_1)(u - b_2)(b_3 - u) \equiv G(u) \), describes periodic motion of a ‘particle’ with co-ordinate \( u \) and time \( \theta \) in the potential \( -G(u) \) (indeed, differentiating (21) with respect to \( u \) we obtain a nonlinear oscillator equation \( u_{\theta\theta} = -dV/du \), where \( V(u) = -G(u) \) is a ‘potential’).

Since \( G(u) > 0 \) for \( u \in [b_2, b_3] \), the ‘particle’ oscillates between the endpoints \( b_2 \) and \( b_3 \).
Appendix. Periodic travelling wave solutions

The period of the oscillations (the wavelength of the travelling wave $u(x - ct)$) is

$$L = \int_{0}^{L} d\theta = 2 \int_{b_2}^{b_3} \frac{du}{\sqrt{2G(u)}} = \frac{2\sqrt{2}K(m)}{(b_3 - b_1)^{1/2}}, \quad (23)$$

where $K(m)$ is the complete elliptic integral of the first kind and $m$ is the modulus,

$$m = \frac{b_3 - b_2}{b_3 - b_1}, \quad 0 \leq m \leq 1. \quad (24)$$

The wavenumber and the frequency of the travelling wave $u(x, t)$ are

$$k = \frac{2\pi}{L}, \quad \omega = kc. \quad (25)$$

Equation (21) is integrated in terms of the Jacobian elliptic cosine function $cn(\xi; m)$ to give

$$u(x, t) = b_2 + (b_3 - b_2) \text{cn}^2 \left( \sqrt{2(b_3 - b_1)}(x - ct - x_0); m \right), \quad (26)$$

where $x_0$ is an initial phase. Solution (26) is often referred to as the cnoidal wave.
Appendix. Periodic travelling wave solutions: two important limits

1. Linear wave, \( m \to 0 \). According to the properties of elliptic functions, when \( m \to 0 \) (\( b_2 \to b_3 \) or, equivalently, \( a \to 0 \)), the cnoidal wave converts into the vanishing amplitude harmonic wave

\[
u(x, t) \approx b_3 - a \sin^2[k_0(x - c_0 t - x_0)], \quad a = b_3 - b_2 \ll 1,
\]

where \( k_0 = k(b_1, b_3, b_3) \), \( c_0 = c(b_1, b_3, b_3) \). Then, using the general expressions for \( k \) and \( c \) one arrives at the relationship

\[
c_0 = 6b_3 - k_0^2,
\]

which agrees with the KdV linear dispersion relation \( \omega(k) = 6ku_0 - k^3 \) for linear waves propagating against the background \( u_0 = b_3 \).

1. Solitary wave, \( m \to 1 \). When \( m \to 1 \) (i.e. \( b_2 \to b_1 \) or, equivalently, \( k \to 0 \) ), the cnoidal wave (26) turns into the solitary wave (soliton)

\[
u_s(x, t) = b_1 + a_s \text{sech}^2[\sqrt{a_s/2}(x - c_s t - x_0)],
\]

whose speed of propagation \( c_s = c(b_1, b_1, b_3) \) is connected with the amplitude \( a_s = b_3 - b_1 \) by the relation

\[
c_s = b_1 + 2a_s.
\]
References

**Taken from the main list of references**


**Additional references**

