Kinetic equation for a soliton gas

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Description of “integrable turbulence”.


A particular important aspect: theory of soliton gases.


Here we consider only strongly integrable systems (like KdV, NLS etc.)
**From \(N\)-solitons/\(N\)-gap potentials to a soliton gas**

**\(N\)-solitons:** two approaches

- **IST:** reflectionless potentials (\(N\)-soliton solutions);
- **Finite-gap theory:** closing all spectral bands in the \(N\)-gap potential leads to the \(N\)-soliton.

**Soliton gas:** \(N \to \infty\); a generalised reflectionless potential (Marchenko) with shift invariant probability measure on it.

- **IST:**
  - **Gurevich, Zybin et. al.** (2000, 2002): stochastic version of the Lax-Levermore approach (KdV and defocusing NLS)
  - **Kotani** (2008): KdV flow on generalized reflectionless potentials (Sato’s Grassmanian approach)

- **Finite-gap theory:** El, Krylov, Molchanov and Venakides (1999): soliton gas as the thermodynamic limit of finite-gap potentials:

\[
N \to \infty, \quad L \to \infty \quad N/L \sim \sum_{j=1}^{N} k_j = O(1)
\]
Assuming the existence of the invariant spectral measure characterising spatially **homogeneous** soliton gas, we want to describe **slow evolution** of this measure for an **inhomogeneous** gas, i.e. to derive **kinetic equation** for solitons.

Similarity with the formulation of the **Whitham modulation theory** (but with the **averaging over ensemble** rather than period!).

**P.D. Lax (1991): The zero dispersion limit, a deterministic analogue of turbulence.**

Hence our interest in the **thermodynamic limit** of the Whitham equations.
Outline

- Kinetic equation for solitons as the thermodynamic limit of the Whitham equations
- Generalised kinetic equation
- Hydrodynamic reductions and integrability
- Conclusions

Also, if time permits;

- Moments of the wave field
- Further research directions
The generating equation for the KdV-Whitham system is (*Flaschka, Forest, McLaughlin 1979*)

\[(dp_N)_t = (dq_N)_x ,\]

where \(dp_N\) and \(dq_N\) are the quasimomentum and quasienergy differentials on the hyperelliptic Riemann surface of genus \(N\):

\[R^2(\lambda) = \prod_{j=1}^{2N+1} (\lambda - \lambda_j) , \quad \lambda \in \mathbb{C}, \quad \lambda_j \in \mathbb{R} .\]

Asymptotic behaviour near \(\lambda = -\infty\):

\[-\lambda \gg 1 : \quad dp_N \sim -\frac{d\lambda}{(-\lambda)^{1/2}} , \quad dq_N \sim (-\lambda)^{1/2} d\lambda .\]
The Whitham equations

The differentials $dp_N$ and $dq_N$ are uniquely defined by the normalisation:

\[
\oint_{\beta_i} dp_N(\lambda) = 0, \quad \oint_{\beta_i} dq_N(\lambda) = 0, \quad i = 1, \ldots, N.
\]

The fundamental wavenumbers $k_j$ and frequencies $\omega_j$ are found as

\[
k_j(\lambda_1, \ldots, \lambda_{2N+1}) = \oint_{\alpha_j} dp_N(\lambda), \quad \omega_j(\lambda_1, \ldots, \lambda_{2N+1}) = \oint_{\alpha_j} dq_N(\lambda),
\]
Whitham equations as the equations for the spectral measure

Assume that the finite-band part of the spectrum \( \lambda \) lies in \([-1, 0]\). We integrate the Whitham system \( \partial_t dp_N(\lambda) = \partial_x dq_N(\lambda) \) on the real line from \(-1\) to \(-\eta^2 \in [-1, 0]\) and take the real part:

\[
\partial_t N_N(-\eta^2) = \partial_x \nu_N(-\eta^2).
\]

Here

\[
N_N(\lambda) = \frac{1}{\pi} \text{Re} \int_{-1}^{\lambda} dp_N(\lambda')
\]

is the integrated density of states (Johnson & Moser 1982), and

\[
\nu_N(\lambda) = \frac{1}{\pi} \text{Re} \int_{-1}^{\lambda} dq_N(\lambda')
\]

– its temporal analog.

Importantly, \( dN_N(-\eta^2) \) is a measure.
Thermodynamic limit

The total density of states

\[ \rho_N = \frac{1}{\pi} \text{Re} \int_{-1}^{0} d\rho_N(\lambda') = \frac{1}{2\pi} \sum_{j=1}^{N} k_j \]

In the thermodynamic limit \( \lim_{N \to \infty} \rho_N = O(1) \).

This is achieved by the following (thermodynamic) spectral scaling

\[ |\text{gap}_j| \sim \frac{1}{\varphi(\eta_j)N} \quad |\text{band}_j| \sim \exp \{-\gamma(\eta_j)N\}, \quad j = 1, \ldots, N \]

where \( \varphi(\eta) \), \( \gamma(\eta) \) are some continuous positive functions on \([0, 1]\).

- Venakides (1989) The continuum limit of theta functions;
- El, Krylov, Molchanov & Venakides (1999) Soliton turbulence as the thermodynamic limit of soliton lattices

Note: \( |\text{band}_j|/|\text{gap}_j| \to 0 \) as \( N \to \infty \) \( \forall j \) i.e. “infinite-soliton” limit.
The thermodynamic limit of the Whitham equations

- The modulation system $\partial_t N_N(-\eta^2) = \partial_x V_N(-\eta^2)$
- In the thermodynamic limit as $N \to \infty$:
  - $dN_N \to \pi f(\eta) d\eta > 0$, $dV_N \to -\pi f(\eta) s(\eta) d\eta$
  - $s(\eta)$ and $f(\eta)$ are related via:

\[
s(\eta) = -4\eta^2 + \frac{1}{\eta} \int_0^1 \ln \left| \frac{\eta - \mu}{\eta + \mu} \right| f(\mu) [s(\eta) - s(\mu)] d\mu, \tag{1}
\]

Now, we postulate that on a larger scale, $\Delta x, \Delta t \gg 1$:

\[
f(\eta) = f(\eta, x, t), \quad s(\eta) = s(\eta, x, t)
\]

Then the modulation system transforms into

\[
f_t = (fs)_x, \tag{2}
\]

Equations (2), (1) form a closed system: the kinetic equation for the KdV soliton gas of \textbf{finite density} \cite{EI2003}
Kinetic equation for solitons: small-density expansion

We replace $s \rightarrow -s$. Now $s$ is the velocity of soliton gas.

\[ f_T + (fs)_x = 0, \quad (1) \]

\[ s(\eta) = 4\eta^2 + \frac{1}{\eta} \int_0^\infty \ln \left| \frac{\eta + \mu}{\eta - \mu} \right| f(\mu)[s(\eta) - s(\mu)]d\mu, \quad (2) \]

The small-density, $\rho = \int_0^\infty fd\eta \ll 1$, expansion of (2), yields

\[ s(\eta) = 4\eta^2 + \frac{1}{\eta} \int_0^\infty \ln \left| \frac{\eta + \mu}{\eta - \mu} \right| f(\mu)[4\eta^2 - 4\mu^2]d\mu + O(\rho^2), \quad (3) \]

– the velocity of a ‘trial’ soliton in a rarefied soliton gas (Zakharov 1971). So Eqs. (1), (2) represent a generalisation of Zakharov’s kinetic equation for a rarefied soliton gas to the case of the gas of finite density.
Main ingredients: (i) **the speed of a free soliton** $S(\eta)$ and (ii) **the phase shift** $\Delta x_{\eta,\mu} = G(\eta, \mu)$ due to the soliton-soliton collision.

Introduce the spectral distribution function $f(\eta) \equiv f(\eta, x, t)$ and the mean speed of a ‘trial’ $\eta$- soliton $s(\eta) \equiv s(\eta, x, t)$.

Then the self-consistent definition of the soliton velocity $s(\eta)$ in a dense soliton gas with the spectral distribution $f(\eta)$ is given by the integral equation

$$s(\eta) = S(\eta) + \int_0^\infty G(\eta, \mu)[s(\eta) - s(\mu)]f(\mu)d\mu,$$

Isospectrality implies the conservation equation for the spectral distribution function $f(\eta, x, t)$:

$$f_t + (sf)_x = 0.$$
Hydrodynamic reductions of the kinetic equation.

\[ f_t + (sf)_x = 0, \quad s(\eta, x, t) = S(\eta) + \int_0^\infty G(\eta, \mu)[s(\eta, x, t) - s(\mu, x, t)]f(\mu, x, t) d\mu \]

(1)

We introduce \( u(\eta, x, t) = \eta f(\eta, x, t) \), \( v(\eta, x, t) = -s(\eta, x, t) \) and consider \( N \)-component ‘cold-gas’ ansatz

\[ u = \sum_{i=1}^{N} u^i(x, t) \delta(\eta - \eta_i), \]

which reduces (1) to a system of \( N \) hydrodynamic conservation laws,

\[ \partial_t u^i = \partial_x (u^i v^i), \quad i = 1, \ldots, N, \]

where the velocities \( v^i = v(\eta_i, x, t) \) and the ‘densities’ \( u^i \) are related via

\[ v^i = \xi_i + \sum_{k=1}^{N} \epsilon_{ik} u^k(v^k - v^i), \quad \epsilon_{ik} = \epsilon_{ki}, \]

\[ \xi_i = S(\eta_i), \quad \epsilon_{ik} = \frac{1}{\eta_i \eta_k} G(\eta_i, \eta_k) > 0, \quad v^i = s(\eta_i). \]
Hydrodynamic reductions: $N = 2$

For $N = 2$ the system of hydrodynamic laws assumes the form

$$
\partial_t u^1 = \partial_x (u^1 v^1), \quad \partial_t u^2 = \partial_x (u^2 v^2)
$$

$$
u^1 = \frac{1}{\epsilon_{12}} \frac{v^2 - \xi_2}{v^1 - v^2}, \quad u^2 = \frac{1}{\epsilon_{12}} \frac{v^1 - \xi_1}{v^2 - v^1}.
$$

Passing to the Riemann invariants we obtain

$$
v^1_t = v^2 v^1_x, \quad v^2_t = v^1 v^2_x. \quad (1)
$$

The system (1) is linearly degenerate, i.e. its characteristic velocities do not depend on the corresponding Riemann invariants.

What about $N > 2$?
Hydrodynamic reductions: $N \geq 3$.

**Theorem** (El, Kamchatnov, Pavlov & Zykov 2008)

*N-component hydrodynamic type system*

$$\partial_t u^i = \partial_x (u^i v^i), \quad i = 1, \ldots, N,$$

$$v^i = \xi_i + \sum_{k=1}^{N} \epsilon_{ik} u^k (v^k - v^i), \quad \epsilon_{ik} = \epsilon_{ki},$$

where $\xi_1, \xi_2, \ldots, \xi_N$ are constants and $\epsilon$ is a constant symmetric matrix, $\epsilon_{ik} = \epsilon_{ki}$, is:

- **diagonalizable**
- **linearly degenerate**,
- **semi-Hamiltonian** *(i.e. integrable - Tsarev 1985, 1991)*,

for any $N$.

**The proof** is based on the theory of integrable linearly degenerate hydrodynamic type systems developed by Pavlov (1987) and Ferapontov (1991).
**Definition.** A diagonal hydrodynamic type system $r^i_t = V^i(r)r^i_x$ is called *linearly degenerate* if $\partial_i V^i = 0 \forall i$. ($\partial_k \equiv \partial/\partial r^k$)

A *semi-Hamiltonian* (i.e. integrable) linearly degenerate hydrodynamic type system $r^i_t = V^i(r)r^i_x$ is characterised by the so-called *Stäkel matrix*

$$
\Delta = \\
\begin{pmatrix}
\phi_1^1(r^1) & \cdots & \phi_1^N(r^N) \\
\vdots & \ddots & \vdots \\
\phi_1^{N-2}(r^1) & \phi_1^{N-1}(r^1) & \phi_N^{N-2}(r^N) \\
1 & \cdots & 1 \\
\end{pmatrix}
$$

where $\phi_k^i(r^k)$ are certain functions, so that (Ferapontov 1991)

$$
V^i(r) = \frac{\det \Delta^{(2)}_i}{\det \Delta^{(1)}_i},
$$

where $\Delta^{(k)}_i$ is the matrix $\Delta$ without $k$-th row and $i$-th column.
It follows from Pavlov (1987) and Ferapontov (1991) that the system of conservation laws

\[ u_t^i = (u^i v^i)_x, \quad v^i = v^i(u(r)) \quad i = 1, \ldots, N \]

is a semi-Hamiltonian linearly degenerate hydrodynamic type system iff the densities \( u^i \) and velocities \( v^i(u) \) admit the representations

\[ u^i = \frac{\det \Delta_i^{(1)}}{\det \Delta} (-1)^{i+1} P_i(r^i), \quad v^i = \frac{\det \Delta_i^{(2)}}{\det \Delta_i^{(1)}} \]

in terms of the Stäkel matrix \( \Delta \) via \( N \) functions \( r^k \); here \( P_i(r^i) \) are arbitrary functions.

For the \( N \) - component hydrodynamic reductions of the kinetic equation it was proved in (El, Kamchatnov, Pavlov & Zykov 2008) that such a parametrization exists for any \( N \).

Hence: integrability of the ‘cold-gas’ hydrodynamic reductions for any \( N \).
The three-component ‘cold-gas’ hydrodynamic reduction of the nonlocal kinetic equation

\[ \partial_t u^i = \partial_x (u^i v^i), \quad i = 1, 2, 3, \]

\[ v^i = \xi_i + \sum_{k=1}^{3} \epsilon_{ik} u^k (v^k - v^i) \quad \epsilon_{ik} = \epsilon_{ki}. \]

has the Riemann invariant representation

\[ \partial_t r^j = V^j(r) \partial_x r^j, \quad j = 1, 2, 3, \]

where

\[ V^1 = \frac{\zeta_2 r^2 - \zeta_3 r^3}{r^2 - r^3}, \quad V^2 = \frac{\zeta_3 r^3 - \zeta_1 r^1}{r^3 - r^1}, \quad V^3 = \frac{\zeta_1 r^1 - \zeta_2 r^2}{r^1 - r^2} \]

\[ \zeta_1 = \frac{\xi_1 \epsilon_{12} - \xi_2 \epsilon_{13}}{\epsilon_{12} - \epsilon_{13}}, \quad \zeta_2 = \frac{\xi_1 \epsilon_{23} - \xi_3 \epsilon_{12}}{\epsilon_{23} - \epsilon_{12}}, \quad \zeta_3 = \frac{\xi_1 \epsilon_{23} - \xi_2 \epsilon_{13}}{\epsilon_{23} - \epsilon_{13}} \]
The Riemann invariants $r^1, r^2, r^3$ are expressed in terms of the densities $u^1, u^2, u^3$ as

\[
\begin{align*}
  r^1 &= \frac{(\epsilon_{12} - \epsilon_{13})(\epsilon_{12}\epsilon_{13}u^1 + \epsilon_{12}\epsilon_{23}u^2 + \epsilon_{13}\epsilon_{23}u^3 + \epsilon_{23})}{[(\xi_3 - \xi_1)\epsilon_{12} + (\xi_1 - \xi_2)\epsilon_{13}]u^1 - (\xi_2 - \xi_3)(\epsilon_{12}u^2 + \epsilon_{13}u^3 + 1)}, \\
  r^2 &= \frac{(\epsilon_{23} - \epsilon_{12})(\epsilon_{12}\epsilon_{13}u^1 + \epsilon_{12}\epsilon_{23}u^2 + \epsilon_{13}\epsilon_{23}u^3 + \epsilon_{13})}{[(\xi_1 - \xi_2)\epsilon_{23} + (\xi_2 - \xi_3)\epsilon_{12}]u^2 - (\xi_3 - \xi_1)(\epsilon_{12}u^1 + \epsilon_{23}u^3 + 1)}, \\
  r^3 &= \frac{(\epsilon_{13} - \epsilon_{23})(\epsilon_{12}\epsilon_{13}u^1 + \epsilon_{12}\epsilon_{23}u^2 + \epsilon_{13}\epsilon_{23}u^3 + \epsilon_{12})}{[(\xi_2 - \xi_3)\epsilon_{13} + (\xi_3 - \xi_1)\epsilon_{23}]u^3 - (\xi_1 - \xi_2)(\epsilon_{13}u^1 + \epsilon_{23}u^2 + 1)}. 
\end{align*}
\]
Conclusions

- The thermodynamic limit of the Whitham equations associated with hyperelliptic Riemann surfaces leads to the kinetic equations for the corresponding soliton gases.

- The original Zakharov (1971) prescription for the determination of the velocity of a trial soliton in a rarefied soliton can be directly extended to the case of the soliton gas of finite density.

- \( N \)-component cold gas hydrodynamic reductions of the generalized kinetic equation for solitons represent linearly degenerate semi-Hamiltonian (integrable) systems of hydrodynamic type for any \( N \).


Wave field moments in a soliton gas: the KdV case

We have obtained that in the thermodynamic limit $d\mathcal{N}_N \to \pi f(\eta, x, t) d\eta$. On the other hand, it is well known that the integrated density of states $\mathcal{N}_N(\lambda)$ is the generating function for the averaged Kruskal integrals $I_k^{(N)}$:

$$\mathcal{N}_N(\lambda) = 2\sqrt{-\lambda} + \sum_{k=0}^{\infty} \frac{I_k^{(N)}}{(-2\lambda)^k}, \quad -\lambda \gg 1,$$

$$I_0^{(N)} = \lim_{L \to \infty} \frac{1}{L} \int_0^L u_N(x) dx, \quad I_1^{(N)} = \lim_{L \to \infty} \frac{1}{L} \int_0^L u_N^2(x) dx, \ldots$$

Then the thermodynamic limit of Kruskal integrals is found as

$$I_k^{(N)} \to \frac{2^{2k+2}}{2k+1} (-1)^{k+1} \int_0^\infty \eta^{2k+1} f(\eta) d\eta, \quad k = 0, 1, 2, \ldots$$
For the two first moments we have
\[
\bar{u}(x, t) = 4 \int_0^\infty \eta f(\eta, x, t) d\eta, \quad \bar{u}^2(x, t) = \frac{16}{3} \int_0^\infty \eta^3 f(\eta, x, t) d\eta
\]

**Important restriction:**
\[
\sigma = \bar{u}^2 - \bar{u}^2 \geq 0 \quad (1)
\]

For instance, for a one-component soliton gas \( f = f_0(x, t) \delta(\eta - \eta_0) \), so
\[
\bar{u} = 4\eta_0 f_0(x, t), \quad \bar{u}^2 = \frac{16}{3} \eta_0^3 f_0(x, t).
\]

i.e. the variance
\[
\sigma = 16f_0\eta_0^2\left(\frac{\eta_0}{3} - f_0\right).
\]

Now one can see that (1) imposes a restriction on the possible density values \( f_0 \) for a one-component soliton gas with a given \( \eta_0 \):
\[
f_0 < f_{cr} = \frac{\eta_0}{3}.
\]

**Critical density for a soliton gas.**
Future research

- **Connection with hydrodynamic chains and 2+1 dispersionless systems.**

  **Motivation:** The *collisionless Boltzmann kinetic equation* for the distribution function \( f(p, x, t) \)

  \[
  \frac{\partial f}{\partial t} + p \frac{\partial f}{\partial x} - \left( \int \frac{\partial f}{\partial x} dp \right) \frac{\partial f}{\partial p} = 0
  \]

  is equivalent to the *Benney moment chain* (Zakharov 1981, Gibbons 1981)

  \[
  \frac{\partial A_n}{\partial t} + \frac{\partial A_{n+1}}{\partial x} + nA_{n-1} \frac{\partial A_0}{\partial x} = 0, \quad A_n = \int p^n fdp
  \]

  and to the *dispersionless KP-II equation* (Kupershmidt & Manin ??, Gibbons 1984, Kodama 1988)

  \[
  (u_t + 6uu_x)_x + u_{yy} = 0.
  \]

- **Construction of physically relevant exact solutions.**

- **PDF, correlation function.**
Appendix 1. N=3: Exact solutions.

1. Similarity solutions

The family of the similarity solutions:

\[ r^i = \frac{1}{t^\alpha} l^i \left( \frac{x}{t} \right), \quad i = 1, 2, 3, \]

is implicitly specified by the algebraic system

\[
\frac{x}{t} = c_1 \zeta_1 (l_1)^\gamma + c_2 \zeta_2 (l_2)^\gamma + c_3 \zeta_3 (l_3)^\gamma, \\
-1 = c_1 (l_1)^\gamma + c_2 (l_2)^\gamma + c_3 (l_3)^\gamma, \\
0 = c_1 (l_1)^{\gamma-1} + c_2 (l_2)^{\gamma-1} + c_3 (l_3)^{\gamma-1},
\]

where \( \gamma = -1/\alpha \) and \( c_1, c_2, c_3 \) are arbitrary constants.
Appendix 1. N=3: Exact solutions.

2. Quasi-periodic solutions

The family of the quasi-periodic (3 periods) solution is implicitly specified by the system

\[
\begin{align*}
\mathbf{x} &= \zeta_1 \int_{r_1} \frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \zeta_2 \int_{r_2} \frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \zeta_3 \int_{r_3} \frac{\xi d\xi}{\sqrt{R_7(\xi)}}, \\
-t &= \int_{r_1} \frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \int_{r_2} \frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \int_{r_3} \frac{\xi d\xi}{\sqrt{R_7(\xi)}}, \\
0 &= \int_{r_1} \frac{d\xi}{\sqrt{R_7(\xi)}} + \int_{r_2} \frac{d\xi}{\sqrt{R_7(\xi)}} + \int_{r_3} \frac{d\xi}{\sqrt{R_7(\xi)}},
\end{align*}
\]

where

\[
R_7(\xi) = \prod_{m=1}^{7} (\xi - E_m),
\]

\(E_1 < E_2 < \cdots < E_7\) are arbitrary real constants.
Appendix 2. Kinetic equation for the focusing NLS soliton gas

The focusing nonlinear Schrödinger (NLS) equation has the form

\[ iu_t + u_{xx} + 2|u|^2 u = 0 \]

Then the kinetic equation for a dense gas of bright NLS solitons is

\[ f_T + (sf)_X = 0, \]

\[ s(\alpha, \gamma) = -4\alpha + \frac{1}{2\gamma} \int_{-\infty}^{\infty} \int_0^{\infty} \ln \left| \frac{\lambda - \mu}{\lambda - \mu} \right|^2 f(\xi, \eta)[s(\alpha, \gamma) - s(\xi, \eta)]d\xi d\eta. \]

Here

\[ f \equiv f(\alpha, \gamma; X, T), \quad s \equiv s(\alpha, \gamma; X, T) \]

\[ \lambda = \alpha + i\gamma, \quad \mu = \xi + i\eta \]
Appendix 2. Kinetic equation for the focusing NLS soliton gas

N=2: collision of two cold soliton gases.
Let \( \rho_1(x, t) = \rho_{10} = \text{constant}_1 \), \( \rho_2(x, t) = \rho_{10} = \text{constant}_2 \).

No continuous solution.

Two **hyperbolic** conservation laws available:

\[
\frac{\partial \rho_1}{\partial t} + \frac{\partial (s_1 \rho_1)}{\partial x} = 0, \quad \frac{\partial \rho_2}{\partial t} + \frac{\partial (s_2 \rho_2)}{\partial x} = 0,
\]

\[
s_1 = s_1(\rho_1, \rho_2) \quad s_2 = s_2(\rho_1, \rho_2)
\]

**Weak solution**: three constant states separated by two strong discontinuities.
A2. Kinetic equation for the focusing NLS soliton gas

N=2: collision of two cold soliton gases.

Jump conditions:

\[-c^- [\rho_1 c - \rho_1^-] + [\rho_1 c s_1 c - \rho_1^- s_1^-] = 0,\]
\[-c^- [\rho_2 c - \rho_2^-] + [\rho_2 c s_2 c - \rho_2^- s_2^-] = 0,\]
\[-c^+ [\rho_1 c - \rho_1^+] + [\rho_1 c s_1 c - \rho_1^+ s_1^+] = 0,\]
\[-c^+ [\rho_2 c - \rho_2^+] + [\rho_2 c s_2 c - \rho_2^+ s_2^+] = 0.\]

As a result, the densities and velocities of the components in the interaction region are

\[\rho_{1c} = \frac{\rho_{10}(1 - \kappa \rho_{20})}{1 - \kappa^2 \rho_{10} \rho_{20}}, \quad \rho_{2c} = \frac{\rho_{20}(1 - \kappa \rho_{10})}{1 - \kappa^2 \rho_{10} \rho_{20}}.\]

\[s_{1c} = 4\alpha \frac{1 - \kappa (\rho_{1c} - \rho_{2c})}{1 - \kappa (\rho_{1c} + \rho_{2c})}, \quad s_{2c} = -4\alpha \frac{1 + \kappa (\rho_{1c} - \rho_{2c})}{1 - \kappa (\rho_{1c} + \rho_{2c})}.\]

The speeds of the ‘shocks’:

\[c^- = -4\alpha \frac{1 + \kappa \rho_{10}}{1 - \kappa \rho_{10}}, \quad c^+ = 4\alpha \frac{1 + \kappa \rho_{20}}{1 - \kappa \rho_{20}}.\]